UC3M
Matemáticas para la Economía II (recuperación)
18 de junio de 2021 **SOLVED**

$\boxed{1}$

Se considera el sistema lineal que consta de dos parámetros $a, b \in \mathbb{R}$.

$$\left\{\begin{array}{rcl} ax + ay + z & = & 1 \\ x + y + az & = & b \\ x + ay + z & = & a \end{array}\right.$$

(a) (15 puntos) Discutir el sistema según los valores de los parámetros a y b.

(b) (5 puntos) Resolver el sistema cuando $a = -1$.

Solución:

(a) We consider the augmented matrix after exchanging the top and the bottom lines.

$$A^* = \left(\begin{array}{ccc|c} 1 & 1 & a & b \\ 1 & a & 1 & a \\ a & a & 1 & 1 \end{array}\right)$$

By Gauss operations $row2 - row1$ and $row3 - a \times row1$ we get

$$\left(\begin{array}{ccc|c} 1 & 1 & a & b \\ 0 & a-1 & 1-a & a-b \\ 0 & 0 & 1-a^2 & 1-ab \end{array}\right)$$

If $a \neq -1$ and $a \neq 1$, then $\text{rank}(A) = \text{rank}(A^*) = 3 =$ number of unknows and from the Rouché-Fröbenius Theorem the system is consistent and determined.

If $a = -1$ and $b = -1$, then $\text{rank}(A) = 2 = \text{rank}(A^*) < 3$ and from the Rouché-Fröbenius Theorem the system is consistent and underdetermined.

If $a = -1$ and $b \neq -1$, then $\text{rank}(A) = 2 < 3 = \text{rank}(A^*)$ and from the Rouché-Fröbenius Theorem the system is inconsistent.

If $a = 1$ and $b = 1$, then $\text{rank}(A) = 1 = \text{rank}(A^*) < 3$ and from the Rouché-Fröbenius Theorem the system is consistent and underdetermined.

If $a = 1$ and $b \neq 1$, then $\text{rank}(A) = 1 < 2 = \text{rank}(A^*)$ and from the Rouché-Fröbenius Theorem the system is inconsistent.

(b) When $a = -1$, the system is consistent only if $b = -1$. In this case the system is underdetermined. The augmented matrix becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and the system is ($z = t$ is taken as parameter)

$$\left.\begin{array}{rcl} x + y - t & = & -1 \\ -2y + 2t & = & 0 \end{array}\right\}$$

The solution are given by vectors $(-1, t, t)$, $t \in \mathbb{R}$.

$\boxed{2}$

Se considera la siguiente matriz de parámetros reales $\alpha \neq 0$ y β.

$$A = \begin{pmatrix} \beta & 0 & 0 \\ 1 & 0 & -\alpha \\ 0 & -\alpha & 0 \end{pmatrix}$$

(a) (10 puntos) ¿Para qué valores de los parámetros $\alpha \neq 0$ y β es la matriz A diagonalizable? Justificar la respuesta.

(b) (10 puntos) Para los valores de los parámetros $\alpha \neq 0$ y β para los que la matriz A es diagonalizable, hallar la matriz P y la matriz diagonal D asociadas a A. Justificar la respuesta.

Nota: ¡La condición $\alpha \neq 0$ es importante para limitar el número de casos!

Solución:

(a) Expanding the determinant $|A - \lambda I|$ by the first row, we obtain

$$\begin{vmatrix} \beta - \lambda & 0 & 0 \\ 1 & -\lambda & -\alpha \\ 0 & -\alpha & -\lambda \end{vmatrix} = (\beta - \lambda)\left(\lambda^2 - \alpha^2\right).$$

Hence, the eigenvalues are $\lambda = \beta$, $\lambda = \alpha$ and $\lambda = -\alpha$. We consider three cases: (i) $\beta = \alpha$; (ii) $\beta = -\alpha$; (iii) $\beta \neq \alpha$ and $\beta \neq -\alpha$. In what follows, remember that $\alpha \neq 0$.

- In case (i), $\lambda = \alpha$ is a double eigenvalue, thus A is diagonalizable iff $\operatorname{rank}(A - \alpha I) = 1$. Since

$$A - \alpha I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -\alpha & -\alpha \\ 0 & -\alpha & -\alpha \end{pmatrix},$$

  has rank 2 because $\alpha \neq 0$, A is not diagonalizable.

- In case (ii), $\lambda = -\alpha$ is a double eigenvalue, thus A is diagonalizable iff $\operatorname{rank}(A + \alpha I) = 1$. Since

$$A + \alpha I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \alpha & -\alpha \\ 0 & -\alpha & \alpha \end{pmatrix},$$

  has rank 2 because $\alpha \neq 0$, A is not diagonalizable.

- In case (iii) A's eigenvalues are distinct. Thus A is diagonalizable.

(b) By the above item, we suppose that $\alpha \neq \beta$ and $\alpha \neq -\beta$. The eigenvalues of A are α, $-\alpha$ and β.

- $S(\alpha)$. Generated by $(0, -1, 1)$.
- $S(-\alpha)$. Generated by $(0, 1, 1)$.
- $S(\beta)$. Generated by $(\alpha^2 - \beta^2, -\beta, \alpha)$.

$\boxed{3}$

(a) (10 puntos) Clasificar la forma cuadrática $Q(x, y, z) = 4x^2 + y^2 + z^2 - 2cxy - 4xz$, donde c es un parámetro real.

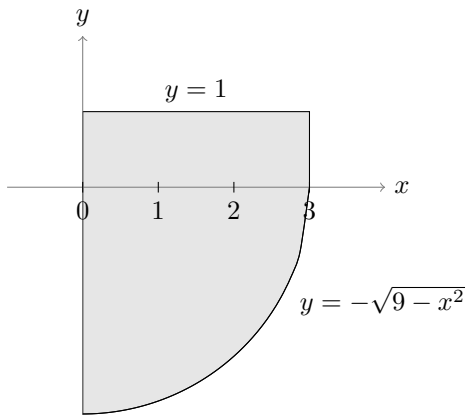
(b) (10 puntos) Representar el conjunto plano $D = \{(x, y) \in \mathbb{R}^2 \,:\, -\sqrt{9 - x^2} \leq y \leq 1, 0 \leq x \leq 3\}$ y calcular la integral doble

$$\iint_D x dx\, dy.$$

Solución:

(a) The matrix associated to Q has leading minors $D_1 = 4$, $D_2 = 4 - c^2$ and $D_3 = -c^2$. Thus, Q is indefinite for all $c \neq 0$ and it is positive semidefinite if $c = 0$.

(b) The set D is the shadow region below.



$$\iint_D x\, dxdy = \int_0^3 x\, dx \int_{-\sqrt{9-x^2}}^1 dy = \int_0^3 \left[x + x\sqrt{9 - x^2}\right] dx = \left[\frac{x^2}{2}\right]_0^3 - \left[\frac{1}{3}(9 - x^2)^{\frac{3}{2}}\right]_0^3 = \frac{9}{2} + \frac{1}{3}(\sqrt{9})^3 = \frac{27}{2}.$$

$\boxed{4}$

(a) (10 puntos) Discutir el carácter de la integral impropia

$$\int_1^e \frac{1}{x \ln x}\, dx.$$

En caso de que la integral sea convergente, hallar su valor.

(b) (10 puntos) Sabiendo que $\lim_{x\to\infty} \dfrac{x}{e^x} = 0$, calcular el valor de la integral

$$\int_2^\infty (x-1)e^{-x}\, dx.$$

Solución:

(a) This is an improper integral of the second kind, since the integrand is not bounded at $x = 1$ ($1/\ln 1 = 1/0 = \infty$.) Note that an antiderivative of the integrand in any interval $(1, b)$ is immediate to obtain: $\ln(\ln x)$. Hence (shortening the notation)

$$\int_1^e \frac{1}{x \ln x}\, dx = [\ln(\ln x)]_1^e = \ln 1 - \ln 0 = 0 - (-\infty) = \infty.$$

The integral does not converge.

(b) It is an improper integral of the second type, since the interval of integration is unbounded (and the integrand is bounded on bounded intervals). Shortening the notation, we have

$$\int_2^\infty (x-1)e^{-x}\, dx = \left[(1-x)e^{-x}\right]_2^\infty + \int_2^\infty e^{-x}\, dx = \frac{1}{e^2} - \left[e^{-x}\right]_2^\infty = \frac{1}{e^2} + \frac{1}{e^2} = \frac{2}{e^2},$$

where we have taken parts $u = x - 1$, $dv = e^{-x}dx$, $du = dx$, $v = -e^{-x}$ and used the hint provided in the question, that is, $xe^{-x} \to 0$ as $x \to \infty$.

$\boxed{5}$

(a) (10 puntos) Se sabe que la sucesión de números reales (x_n) satisface las siguientes propiedades:

- $x_1 = \frac{1}{2}$;
- $x_{n+1} = \sqrt{2x_n + 3}$;
- Converge hacia el número real L, es decir, $\lim_{n\to\infty} x_n = L$.

Hallar el valor del límite L.

(b) (10 puntos) Estudiar el carácter de la serie

$$\sum_{n=2}^{\infty} (a^{\frac{1}{2(n-1)}} - a^{\frac{1}{2n}}), \quad \text{donde } a > 0.$$

En el caso de que la serie sea convergente, encontrar su valor.
Pista: Notar que se trata de una serie telescópica.

Solución:

(a) Since the sequence (x_n) is convergent, the sequence $\sqrt{2x_n+3}$ is also convergent. Moreover

$$\lim_{n\to\infty} \sqrt{2x_n+3} = \sqrt{2\lim_{n\to\infty} x_n + 3} = \sqrt{2L+3},$$

thus

$$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \sqrt{2\lim_{n\to\infty} x_n + 3} = \sqrt{2L+3}.$$

Hence, L satisfies $L = \sqrt{2L+3}$, or $L^2 = 2L+3$, $L^2 - 2L - 3 = 0$. Solving this quadratic equation we get two values, $L = -1$ and $L = 3$. But (x_n) is of positive terms, thus the limit cannot be -1. Thus the limit is $L = 3$.

(b) If $a = 1$, then the series is the null series. Suppose that $a > 0$ is not equal to 1. The series is telescoping, where consecutive terms from the second one cancel out. Hence, considering the partial sum of n terms we get

$$S_n = a^{\frac{1}{2}} \cancel{- a^{\frac{1}{4}} + a^{\frac{1}{4}}} \cancel{- a^{\frac{1}{6}} + a^{\frac{1}{6}}} - \cdots \cancel{- a^{\frac{1}{2(n-1)}} + a^{\frac{1}{2(n-1)}}} - a^{\frac{1}{2n}},$$

that is, $S_n = \sqrt{a} - a^{\frac{1}{2n}}$. Since $\lim_{n\to\infty} a^{\frac{1}{2n}} = a^{\frac{1}{\infty}} = a^0 = 1$, the sum of the series is $\sqrt{a} - 1$ for all $a > 0$.