MATHEMATICS FOR ECONOMICS II. ECONOMICS
Extraordinary call, June 2020

Problem 1

(a) (4 pts.) Compute the rank of the matrix

$$A = \begin{pmatrix} 2x & x & 3x \\ 6x & 1+x & 0 \\ -2x & x-1 & 4x-4 \end{pmatrix}$$

depending on the values of x.

(b) (6 pts.) Compute the eigenvectors and eigenvectors of the matrix

$$B = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Say whether B is diagonalizable.
(Hint: use Ruffini's rule to find the eigenvalues).

Problem 2

(a) (4 pts.) Compute the integral

$$\int \frac{\cos x}{1+\sin x}\, dx.$$

(b) (6 pts.) Compute the volume of the solid defined under the graph of the function $z = xe^{x^2-y}$ and which basis is the region

$$A = \{(x,y) \,:\, 0 \le x \le 1,\ x^2 \le y \le 2x^2\}.$$

Problem 3

Consider the function $F : \mathbb{R} \to \mathbb{R}$

$$F(t) = \int_{-2t}^{2t} e^{-x^2}\, dx.$$

(a) (2 pts.) Show that F is an odd function, that is, $F(-t) = -F(t)$ for all t.

(b) (2 pts.) Find $F(0)$.

(c) (2 pts.) Calculate the value of the following limits.

$$\lim_{t\to\infty} F(t), \qquad \lim_{t\to-\infty} F(t).$$

(Hint: $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.)

(d) (4 pts.) Study the intervals where F is monotone and the intervals where F is concave or convex.

Problem 4

(a) (3 pts.) Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ with general term

$$a_n = (-1)^n \frac{n}{3n-1}$$

has no limit.

(b) (3 pts.) Consider the series

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{\ln n}}$$

Does it converge?

(c) (3 pts.) Does the series defined in part (b) above converge absolutely or conditionally?
Hint: use the integral criterion.

Solutions

(a) The determinant of A is

$$\begin{vmatrix} 2x & x & 3x \\ 6x & 1+x & 0 \\ -2x & x-1 & 4x-4 \end{vmatrix} = \begin{vmatrix} 2x & x & 3x \\ 0 & 1-2x & -9x \\ 0 & 2x-1 & 7x-4 \end{vmatrix} = \begin{vmatrix} 2x & x & 3x \\ 0 & 1-2x & -9x \\ 0 & 0 & -2x-4 \end{vmatrix} = -4x(1-2x)(x+2).$$

(Operations: first equality, $\text{row}_3+\text{row}_1$, $\text{row}_2\text{-}3\times\text{row}_1$; in the second one, $\text{row}_3+\text{row}_2$).

Hence the rank of A is 3 if $x \neq 0$, $x \neq \frac{1}{2}$ and $x \neq -2$. Let us see that the rank is 2 for $x = 0$, $x = \frac{1}{2}$ or $x = -2$.

When $x = 0$, the echelon matrix of A becomes

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

which has rank 2.

When $x = \frac{1}{2}$, the echelon matrix of A becomes

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & \frac{-9}{2} \\ 0 & 0 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & \frac{-9}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

which is also of rank 2.

When $x = -2$, the echelon matrix of A becomes

$$\begin{pmatrix} -4 & -2 & -6 \\ 0 & 5 & 18 \\ 0 & 0 & 0 \end{pmatrix}$$

which is also of rank 2.

(b) The characteristic polynomial of B is $p(\lambda) = (1-\lambda)^3 - 3(1-\lambda) - 2$. Expanding it, we get

$$p(\lambda) = -\lambda^3 + 3\lambda^2 - 4.$$

Integer roots, if exist, divide the independent term, hence they can be ± 1, ± 2 and ± 4. Applying Ruffini's rule with $\lambda = -1$, the polynomial factor as follows: $-(1+\lambda)(\lambda^2 - 4\lambda + 4) = -(1+\lambda)(\lambda - 2)^2$. Thus, the roots are $\lambda = -1$ (simple) and $\lambda = 2$ (double), which are also the eigenvalues of B.

To find the eigenvectors, we have to solve the systems $(B+I)u = 0$ and $(B-2I)u = 0$. The first system becomes

$$\begin{aligned} 2x - y - z &= 0 \\ -x + 2y - z &= 0 \\ -x - y + 2z &= 0 \end{aligned}$$

and solving it we get $x = y = z$, so $(1,1,1)$ is an eigenvector associated to $\lambda = -1$. The second linear system becomes

$$x + y + z = 0,$$

so $(-1,1,0)$ and $(0,-1,1)$ are eigenvectors associated to $\lambda = 2$.

Matrix B is diagonalizable, since it has three independent eigenvectors.

Problem 2

(a) $\int \frac{\cos x}{1+\sin x}\, dx = \ln(1+\sin x) + C.$

(b)

$$\begin{aligned}
V &= \iint_A xe^{x^2-y}\,dxdy = \int_0^1 x\,dx \int_{x^2}^{2x^2} e^{x^2-y}\,dy = \int_0^1 -xe^{x^2-y}\Big|_{x^2}^{2x^2}\,dx \\
&= \int_0^1 -x(e^{-x^2}-1)\,dx = \int_0^1 x\,dx - \int_0^1 xe^{-x^2}\,dx = \frac{1}{2} + \frac{1}{2}e^{-x^2}\Big|_0^1 \\
&= \frac{1}{2} + \frac{1}{2e} - \frac{1}{2} = \frac{1}{2e}.
\end{aligned}$$

Problem 3

(a) $F(-t) = \int_{2t}^{-2t} e^{-x^2}\,dx = -\int_{-2t}^{2t} e^{-x^2}\,dx = -F(t)$, hence F is odd. We have used that the integral changes sign when the order of the interval of integration changes.

(b) $F(0) = \int_0^0 e^{-x^2}\,dx = 0$, since the integral is 0 when the interval of integration is degenerated.

(c) We know (hint) that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, hence $\lim_{t\to\infty} F(t) = \sqrt{\pi}$ and

$$\lim_{t\to-\infty} F(t) = \int_{\infty}^{-\infty} e^{-x^2}\,dx = -\int_{-\infty}^{\infty} e^{-x^2}\,dx = -\sqrt{\pi}.$$

(d) By Leibniz's Theorem, the derivative of F is

$$F'(t) = 2e^{-4t^2} - (-2)e^{-4t^2} = 4e^{-4t^2} > 0, \quad \text{for all } t.$$

Thus, F is strictly increasing. The second derivative of f is

$$F''(t) = -32te^{-4t^2},$$

positive when $t < 0$ and negative when $t > 0$. Hence, F is strictly convex in $(-\infty, 0]$ and strictly concave in $[0, \infty)$.

Problem 4

(a) The subsequence of even terms, $n = 2k$, is given by

$$\frac{(-1)^{2k}\cdot 2k}{3(2k)-1} = \frac{2k}{6k-1}.$$

This subsequence converges to $\frac{1}{3}$.

The subsequence for the odd terms, $n = 2k+1$, is given by

$$\frac{(-1)^{2k+1}\cdot(2k+1)}{3(2k+1)-1} = \frac{-(2k+1)}{6k+2}.$$

This subsequence converges to $\frac{-1}{3}$.

Since there are two subsequences that converge to different limits, there is no limit of the sequence.

(b) It is an alternating series that satisfies

- $a_n > a_{n+1}$, $n =, 1, 2, \ldots$,
- $\lim_{n\to\infty} a_n = 0$,

with $a_n = \frac{1}{n\sqrt{\ln n}}$. By the Leibniz criterion it converges.

(c) We apply the integral test to the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$. The integral

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,dx = 2\sqrt{\ln x}\Big|_2^{\infty}$$

diverges. Hence, the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ is divergent.

We conclude that the series $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{\ln n}}$ converges conditionally.