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IV.2 PARAMETRIC INTEGRALS

1. Functions defined by integrals

Definition 1.1. Let $A \subseteq \mathbb{R}$ and let $f : A \times [a,b] \to \mathbb{R}$. Suppose that the function $f_\lambda : [a,b] \to \mathbb{R}$ defined by

$$f_\lambda(x) = f(\lambda, x)$$

is integrable for all $\lambda \in A$. We say that

$$\int_a^b f_\lambda(x)dx = \int_a^b f(\lambda, x)dx$$

is a parametric integral with fixed limits of integration.

Remark 1.2. Note that $F(\lambda) = \int_a^b f(\lambda, x)dx$ defines a function from A to $\mathbb{R}$.

Definition 1.3. Let $A \subseteq \mathbb{R}$. Let the functions $a, b : A \to \mathbb{R}$ be such that $a(\lambda) \le b(\lambda)$ for all $\lambda \in A$. Consider the set

$$S(a,b) = \{(\lambda, x) \in A \times [a,b] \,:\, a(\lambda) \le x \le b(\lambda)\},$$

and the function $f : S(a,b) \to \mathbb{R}$. Suppose that the function $f_\lambda : [a(\lambda), b(\lambda)] \to \mathbb{R}$ defined by $f_\lambda(x) = f(\lambda, x)$ is integrable for all $\lambda \in A$. We say that

$$\int_{a(\lambda)}^{b(\lambda)} f_\lambda(x)dx = \int_{a(\lambda)}^{b(\lambda)} f(\lambda, x)dx$$

is a parametric integral with variable limits of integration.

Remark 1.4. Note that $F(\lambda) = \int_{a(\lambda)}^{b(\lambda)} f(\lambda, x)dx$ defines a function from A to $\mathbb{R}$.

2. Continuity of the parametric integrals

Theorem 2.1. *Let $A \subseteq \mathbb{R}$ be a compact set and let $f : A \times [a,b] \to \mathbb{R}$ be continuous. Then the function*

$$F(\lambda) = \int_a^b f(\lambda, x)dx$$

is continuous.

Theorem 2.2. *Let $A \subseteq \mathbb{R}$. Let the functions $a, b : A \to \mathbb{R}$ be continuous, such that $a(\lambda) \leq b(\lambda)$ for all $\lambda \in A$. Consider the set*

$$S(a,b) = \{(\lambda, x) \in A \times [a,b] \,:\, a(\lambda) \leq x \leq b(\lambda)\},$$

and the continuous function $f : S(a,b) \to \mathbb{R}$. Then the function $F : A \to \mathbb{R}$ defined by

$$F(\lambda) = \int_{a(\lambda)}^{b(\lambda)} f(\lambda, x)dx$$

is continuous.

Remark 2.3. The importance of the theorems above is that, under the hypotheses, to exchange the limit with the integral sign is allowed. Let $\lambda_0 \in A$. Then

$$\lim_{\lambda \to \lambda_0} \int_a^b f(\lambda, x)dx = \int_a^b \lim_{\lambda \to \lambda_0} f(\lambda, x)dx,$$
$$\lim_{\lambda \to \lambda_0} \int_{a(\lambda)}^{b(\lambda)} f(\lambda, x)dx = \int_{a(\lambda_0)}^{b(\lambda_0)} \lim_{\lambda \to \lambda_0} f(\lambda, x)dx$$

3. Differentiation under the integral sign

Theorem 3.1 (Leibniz's Rule)**.** *Let $A \subseteq \mathbb{R}$ be an open set. Let $f : A \times [a,b] \to \mathbb{R}$ be continuous and such that $\frac{\partial f}{\partial \lambda}(\lambda, x)$ exists and is continuous. Then the function $F : A \to \mathbb{R}$ defined by*

$$F(\lambda) = \int_a^b f(\lambda, x)dx$$

is derivable in A and the derivative is

$$F'(\lambda) = \frac{\partial}{\partial \lambda} \int_a^b f(\lambda, x)dx = \int_a^b \frac{\partial f}{\partial \lambda}(\lambda, x)dx.$$

Theorem 3.2 (Generalized Leibniz's Rule)**.** *Let $A \subseteq \mathbb{R}$ be an open set. Let the functions $a, b : A \to \mathbb{R}$ be of class C^1 in A, such that $a(\lambda) \leq b(\lambda)$ for all $\lambda \in A$. Consider the set*

$$S(a,b) = \{(\lambda, x) \in A \times [a,b] \,:\, a(\lambda) \leq x \leq b(\lambda)\},$$

and an open set U such that $S(a,b) \subseteq U$. Let the function $f : U \to \mathbb{R}$ be continuous, such that $\frac{\partial f}{\partial \lambda}(\lambda, x)$ exists and is continuous. Then the function $F : A \to \mathbb{R}$ defined by

$$F(\lambda) = \int_{a(\lambda)}^{b(\lambda)} f(\lambda, x)dx$$

is derivable in A and the derivative is

$$\begin{aligned} F'(\lambda) &= \frac{\partial}{\partial\lambda}\int_{a(\lambda)}^{b(\lambda)} f(\lambda,x)dx \\ &= \int_{a(\lambda)}^{b(\lambda)} \frac{\partial f}{\partial\lambda}(\lambda,x)dx + f(\lambda,b(\lambda))b'(\lambda) - f(\lambda,a(\lambda))a'(\lambda). \end{aligned}$$

4. Gamma and Beta functions

Definition 4.1 (Gamma function)**.** The gamma function $\Gamma : (0,\infty) \to \mathbb{R}$ is defined as

$$\Gamma(p) = \int_0^\infty e^{-x}x^{p-1}dx.$$

Remark 4.2. Note that Γ is an improper integral, which converges for all $p > 0$.

4.1. Properties of Γ.

(1) $\Gamma(1) = 1$.

(2) $\Gamma(p+1) = p\Gamma(p)$, for all $p > 0$.

(3) $\Gamma(n) = (n-1)!$, for all $p = 1, 2, \ldots$.

(4) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(5) $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}$, for all $0 < p < 1$.

Definition 4.3 (Beta function)**.** The beta function $\beta : (0,\infty)\times(0,\infty) \to \mathbb{R}$ is defined as

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

Remark 4.4. Note that β is an improper integral, which converges for all $p > 0$, $q > 0$.

4.2. Properties of β.

(1) $\beta(p,q) = \beta(q,p)$, for all $p > 0$ and $q > 0$.

(2) $\beta(p,q) = 2\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} dx$, for all $p > 0$ and $q > 0$.

(3) $\beta(p,q) = \frac{\Gamma(p)}{\Gamma(q)}\Gamma(p+q)$, for all $p > 0$ and $q > 0$.

(4) $\beta(p,q) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$, for all $p, q = 1, 2, \ldots$.