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IV.1 MULTIPLE INTEGRALS

The Multiple integral of Riemann is a straightforward extension of the simple integral.

1. Integration in intervals of $\mathbb{R}^p$.

Definition 1.1. A compact interval of $\mathbb{R}^p$ is a set $I = [a_1, b_1] \times \cdots \times [a_p, b_p]$, where each $[a_i, b_i]$ is a compact interval of $\mathbb{R}$, for $i = 1, \ldots, m$.

The measure of the compact interval I is $\mu(I) = (b_1 - a_1) \cdots (b_p - a_p)$.

A finite collection $\mathcal{P} = \{I_1, \ldots, I_n\}$ of compact intervals is a partition of I if (i) $I = I_1 \cup \cdots \cup I_n$ and (ii) $\overset{\circ}{I_i} \cap \overset{\circ}{I_j} = \emptyset$ for $i \neq j$.

Given two partitions $\mathcal{P}'$ and $\mathcal{P}$ of I, $\mathcal{P}'$ is finer than $\mathcal{P}$ if every interval of $\mathcal{P}'$ is contained in some interval of $\mathcal{P}$.

The diameter of the partition $\mathcal{P}$ of I is the larger of the lengths $b_i - a_i$, for $i = 1, \ldots, n$, and will denoted by $|\mathcal{P}|$.

Definition 1.2. Let $f : I \to \mathbb{R}$ be a bounded function defined in the compact interval $I \subseteq \mathbb{R}^p$. Given a partition $\mathcal{P} = \{I_1, \ldots, I_n\}$, let

$$m_i = \inf f(I_i), \qquad M_i = \sup f(I_i)$$

and let $\mu(I_i)$ be the measure of I_i, for $i = 1, \ldots, n$. The lower and the upper (Darboux) sums of f in the partition $\mathcal{P}$ are

$$s(f, \mathcal{P}) = \sum_{i=1}^{p} m_i \mu(I_i), \qquad S(f, \mathcal{P}) = \sum_{i=1}^{p} M_i \mu(I_i),$$

respectively.

Proposition 1.3 (Properties of Darboux sums)**.** *Let $f : I \to \mathbb{R}$ be a bounded function defined in the compact interval $I \subseteq \mathbb{R}^p$. Let*

$$m = \inf f(I), \qquad M = \sup f(I).$$

Let $\mathcal{P}$, $\mathcal{P}'$ two partitions of I.

(1) $m\mu(I) \leq s(f, \mathcal{P}) \leq S(f, \mathcal{P}) \leq M\mu(I)$.

(2) If $\mathcal{P}'$ is finer than $\mathcal{P}$, then

$$s(f, \mathcal{P}) \leq s(f, \mathcal{P}') \leq S(f, \mathcal{P}') \leq S(f, \mathcal{P}).$$

(3) $s(f,\mathcal{P}') \le S(f,\mathcal{P})$.

(4) The sets

$$\{s(f,\mathcal{P}) \,:\, \mathcal{P} \text{ is a partition of } I\} \text{ and } \{S(f,\mathcal{P}) \,:\, \mathcal{P} \text{ is a partition of } I\}$$

are bounded.

Definition 1.4. Let $f : I \to \mathbb{R}$ be a bounded function defined in the compact interval $I \subseteq \mathbb{R}^p$.

The lower integral of f in I is defined as the number

$$L\int_I f = \sup\{s(f,\mathcal{P}) \,:\, \mathcal{P} \text{ is a partition of } I\}.$$

The upper integral of f in $[a,b]$ is defined as the number

$$U\int_I f = \inf\{S(f,\mathcal{P}) \,:\, \mathcal{P} \text{ is a partition of } I\}.$$

Proposition 1.5. *Let $f : I \to \mathbb{R}$ be a bounded function defined in the compact interval $I \subseteq \mathbb{R}^p$. Then*

$$L\int_I f \le U\int_I f.$$

Definition 1.6. Let $f : I \to \mathbb{R}$ be a bounded function defined in the compact interval $I \subseteq \mathbb{R}^p$.

We say that the function is Riemann integrable (or simply integrable) in the interval $I \subseteq \mathbb{R}^p$ iff

$$L\int_I f = U\int_I f.$$

In this case, this number is the integral of f in I (or defined integral of f in I) and is denoted

$$\int_I f, \quad \int_I f(x)dx = \int \overset{p}{\cdots} \int_{[a_1,b_1]\times\cdots\times[a_p,b_p]} f(x_1,\dots,x_p)dx_1\cdots dx_p.$$

Not every bounded function is integrable.

Theorem 1.7. *Let $f : I \to \mathbb{R}$ be a bounded function defined in the compact interval $I \subseteq \mathbb{R}^p$. Thenf is integrable iff for all $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of I such that*

$$S(f,\mathcal{P}) - s(f,\mathcal{P}) < \varepsilon.$$

Proposition 1.8. *If f is continuous in I, then f is integrable in I.*

This result admits a useful generalization.

Proposition 1.9. *If f is bounded in I and has a finite number of discontinuities, then f is integrable in I.*

2. Properties of the Riemann Integral

Proposition 2.1 (Properties of the integral)**.** Let $f, g : I \to \mathbb{R}$ be integrable, where $I \subseteq \mathbb{R}^p$ is a compact interval. Let $\alpha \in \mathbb{R}$.

(1) *Linearity.*

(a) $\int_I (f+g) = \int_I f + \int_I g$.

(b) $\int_I \alpha f = \alpha \int_I f$.

(2) *Monotonicity.*

$$f(x) \geq g(x) \text{ for all } x \in I, \text{ implies } \int_I gf \geq \int_I g.$$

(3) $\left| \int_I f(x)dx \right| \leq \int_I |f(x)|dx$.

(4) *Additivity with respect to the interval.* Let $\{I_1, \dots, I_n\}$ be a partition of I. Then f is integrable in I if and only if f is integrable in $I_1, \dots, I_n$, and in this case

$$\int_I f = \int_{I_1} f + \cdots + \int_{I_n} f.$$

Proposition 2.2 (Theorem of the mean)**.** *Let $f : I \to \mathbb{R}$ be a bounded function defined in the compact interval $I \subseteq \mathbb{R}^p$.*

(1) If f is integrable in I and if m and M are lower and upper bounds of f in I, respectively (they could be the infimum and the supremum), then there is $\alpha \in [m, M]$ such that

$$\int_I f(x) = \alpha\mu(I).$$

(2) If f is continuous in I, then there exists $c \in I$ such that

$$\int_I f(x) = f(c)\mu(I).$$

Remark 2.3. Let $f : I \to \mathbb{R}$ be a bounded function defined in the compact interval $I \subseteq \mathbb{R}^2$ such that $f(x, y) \geq 0$ for all $(x, y) \in I$. Let the hypograph of f

$$H(f) = \{(x, y, z) \in \mathbb{R}^3 \,:\, (x, y) \in I, 0 \leq z \leq f(x, y)\}.$$

The lower (uuper) Darboux sum $s(f, \mathcal{P})$ $(S(f, \mathcal{P}))$ is the sum of volumes of rectangular prisms with basis I_i and height m_i (M_i). Hence, the lower (upper) Darboux sum is an underestimation (overestimation) of the volume of the set $H(F)$. Since both Darboux sums tend to the same number when the diameter of the partitions tend to zero, we say that the integral of f in I is the volume of $H(f)$.

3. Iterated integrals

In this section we show how the integral of a function $f : I \to \mathbb{R}$ on a compact interval $I = [a_1, b_1] \times \cdots \times [a_p, b_p]$ can be expressed as p simple integrals in the intervals $[a_1, b_1], \ldots, [a_p, b_p]$.

Remark 3.1. Given $x = (x_1, \ldots, x_p)$, and $i \in \{1, \ldots, p\}$, x_{-i} denotes the vector of dimension $p-1$ obtained from x after eliminating the coordinate x_i. In the same way, for a compact interval $I = [a_1, b_1] \times \cdots \times [a_p, b_p]$, I_{-i} denotes the Cartesian product of all the individual intervals that form I, except the interval $[a_i, b_i]$. We will write $x = (x_i|x_{-i})$ and $I = (I_i|I_{-i})$.

For instance, if $x = (0, -1, 5)$, then $x_{-1} = (-1, 5)$, $x_{-2} = (0, 5)$ and $x_{-3} = (0, -1)$. If $I = [0, 1] \times [-2, -1]$, then $I_1 = [0, 1]$ and $I_{-1} = [-2, -1]$.

Theorem 3.2 (Fubini's Theorem in compact intervals)**.** *Let $f : I \to \mathbb{R}$ be bounded in I, and suppose that for all $x_{-i} \in I_{-i}$, the function $x_i \mapsto f(x_i|x_i)$ is integrable in $[a_i, b_i]$, for all $i = 1, \ldots, p$. Then $f : I \to \mathbb{R}$ is integrable in I and*

$$\int_I f(x_1, \ldots, x_p) dx_1 \ldots dx_p = \int_{a_1}^{b_1} dx_1 \left(\int_{a_2}^{b_2} dx_2 \left(\cdots \left(\int_{a_p}^{b_p} f(x_1, \ldots, x_p) dx_p \right) \right) \right).$$

Remark 3.3. The theorem assert that the order of computation of the iterated integrals does not matter to find the integral. In the case $p = 2$, the theorem means that

$$\iint_I f(x, y) dx dy = \int_{a_1}^{b_1} dx \int_{a_2}^{b_2} f(x, y) dy = \int_{a_2}^{b_2} dy \int_{a_1}^{b_1} f(x, y) dx.$$

Example 3.4. Calculate

$$\iint_I e^{-xy} x dx dy, \qquad I = [0, 1] \times [-1, 0].$$

Solution: Applying Fubini's Theorem, we can compute the integral by iterated integration.

$$\iint_I e^{-xy} x dx dy = \int_0^1 dx \int_{-1}^0 e^{-xy} x dy = \int_0^1 \left(-e^{-xy} \right) \Big|_{-1}^0 dx = \int_0^1 (e^x - 1) dx = e^x - x \Big|_0^1 = e - 2.$$

Also,

$$\iint_I e^{-xy} x dx dy = \int_{-1}^0 dy \int_0^1 e^{-xy} x dx.$$

The inner integral is not immediate. It can be calculated by using parts.

Theorem 3.5 (Fubini's Theorem in simple regions of $\mathbb{R}^2$). *(1) (type 1 regions). Let φ_1, φ_2 be two continuous functions in $[a,b]$ such that $\varphi_1(x) \leq \varphi_2(x)$ for all $x \in [a,b]$. Let the compact set*

$$S = \{(x,y) \,:\, a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$$

If f is integrable in S, then

$$\int_S f(x,y)dxdy = \int_a^b dx \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right).$$

(2) (type 2 regions) Let ϕ_1, ϕ_2 be two continuous functions in $[c,d]$ such that $\phi_1(y) \leq \phi_2(y)$ for all $y \in [c,d]$. Let the compact set

$$T = \{(x,y) \,:\, c \leq y \leq d, \phi_1(y) \leq x \leq \phi_2(y)\}$$

If f is integrable in T, then

$$\int_T f(x,y)dxdy = \int_c^d dy \left(\int_{\phi_1(y)}^{\phi_2(y)} f(x,y)dx \right).$$

Example 3.6. Calculate

$$\iint_S (x^2+y)dxdy, \qquad S = \{(x,y) \,:\, -1 \leq x \leq 1, x^2 \leq y \leq 1\}.$$

Solution:

$$\begin{aligned} \iint_S (x^2+y)dxdy &= \int_{-1}^1 dx \int_{x^2}^1 (x^2+y)dy = \int_{-1}^1 \left(x^2 y + \frac{y^2}{2} \right)\Big|_{x^2}^1 dx \\ &= \int_{-1}^1 \left[\left(x^2 + \frac{1}{2} \right) - \left(x^4 + \frac{x^4}{2} \right) \right] dx = \frac{16}{15} \end{aligned}$$

Example 3.7. Calculate $\iint_A f(x,y)dxdy$, where A is the triangular region of vertex $(0,0)$, $(1,0)$ and $(1,1)$ and $f(x,y) = e^{\frac{x}{y}}$ if $y > 0$, $f(x,0) = 0$ for all $0 \leq x \leq 1$.

Solution:

Note that f is continuous in $A - \{(0,0)\}$. Since that there is only a point of discontinuity in the region A, f is integrable.

Note that $A = \{(x,y) \,:\, 0 \leq x \leq 1, 0 \leq y \leq x\} = \{(x,y) \,:\, 0 \leq y \leq 1, y \leq x \leq 1\}$

First method: fix x and integrate firstly in y and then in x.

$$\iint_A e^{\frac{y}{x}} dxdy = \int_0^1 dx \int_0^x e^{\frac{y}{x}} dy = \int_0^1 x e^{\frac{y}{x}}\Big|_0^x dx = \int_0^1 \frac{x^2}{2}(e-1)dx = \frac{e-1}{2}$$

Second method: fix y and integrate firstly in x and then in y.

$$\iint_A e^{\frac{y}{x}} dxdy = \int_0^1 dy \int_y^1 e^{\frac{y}{x}} dx$$

is impossible to compute

4. Areas and volumes

Definition 4.1. Let φ_1, φ_2 be two continuous functions in $[a, b]$ such that $\varphi_1(x) \leq \varphi_2(x)$ for all $x \in [a, b]$. Let the compact set

$$S = \{(x, y) : a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}.$$

The area of S is

$$\iint_S dxdy = \int_a^b dx \left(\int_{\varphi_1(x)}^{\varphi_2(x)} dy \right) = \int_a^b (\varphi_2(x) - \varphi_1(x))\, dx.$$

Definition 4.2. Let ϕ_1, ϕ_2 be two continuous functions in $[c, d]$ such that $\phi_1(y) \leq \phi_2(y)$ for all $y \in [c, d]$. Let the compact set

$$T = \{(x, y) : c \leq y \leq d, \phi_1(y) \leq x \leq \phi_2(y)\}.$$

The area of T is

$$\iint_T dxdy = \int_c^d dy \left(\int_{\phi_1(y)}^{\phi_2(y)} dx \right) = \int_c^d (\phi_2(y) - \phi_1(y))\, dy.$$

Definition 4.3. Let A be a subset of $\mathbb{R}^2$ of type 1 or type 2, that is $A = S$ or $A = T$. Let $f : A \to \mathbb{R}_+$ be integrable and nonnegative. The volume of the solid of $\mathbb{R}^3$ limited by the plane $z = 0$, the cylinder generated by the generatrix parallel to the vertical axis z and directrix the boundary of the set A, and the surface of equation $z = f(x, y)$ is

$$\iint_A f(x, y) dxdy.$$

5. Change of variable

The formula of change of variable in simple integrals is an straightforward application of the chain rule. The generalization of the formula to two or more variables is not easy. We will establish only the formula for double integrals.

Theorem 5.1. *Let D be a subset of $\mathbb{R}^2$ which is of type 1 or of type 2. Let U be an open set such that $D \subseteq U$, and let $g : U \to \mathbb{R}^2$ be a transformation of class C^1, one to one and such that the Jacobian of g is non null for all $(x, y) \in \mathring{D}$.*

Let $f : A \to \mathbb{R}$ be a continuous function in $A = g(D)$. Then

$$\iint_A f(x,y)dxdy = \iint_D f(g(u,v))|\det Jg(u,v)|dudv.$$

Remark 5.2. The transformation g is the change of variable. It is often used for g the notation $(x, y) = g(u, v)$, where

$$x = x(u,v),$$
$$y = y(u,v),$$

$(u, v) \in U$. The Jacobian matrix of g is the matrix

$$Jg(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

Note that the Jacobian determinant enters the integral in absolute value.

Example 5.3. Calculate the integral

$$\iint_A (2y - x)dxdy,$$

where A is the region of the plane enclosed by the lines $y = x + 1$, $y = x + 5$, $y = -x + 1$ and $y = -x + 3$.

Solution: Let the change of variables $u = y - x$ and $v = y + x$. Then $x = \frac{1}{2}(v - u)$ and $y = \frac{1}{2}(u + v)$. The transformation is thus $g(u, v) = (\frac{1}{2}(v - u), \frac{1}{2}(u + v))$. Note that if we let $D = [1, 5] \times [1, 3]$, then $g(D) = A$. Moreover, g is clearly one-to-one. The Jacobian of g is

$$\det Jg(u,v) = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

Finally, note that $2y - x = u + v - \frac{1}{2}(v-u) = \frac{3}{2}u + \frac{1}{2}v$, hence

$$\begin{aligned}
\iint_A (2y-x)dxdy &= \iint_D \frac{1}{2}\left(\frac{3}{2}u + \frac{1}{2}v\right) dudv \\
&= \frac{3}{4}\iint_D ududv + \frac{1}{4}\iint_D vdudv \\
&= \frac{3}{4}\int_1^5 2udu + \frac{1}{4}\int_1^3 4vdv \\
&= \frac{3}{4}u^2\Big|_1^5 + \frac{1}{2}v^2\Big|_1^3 \\
&= \frac{3}{4}(25-1) + \frac{1}{2}(9-1) = 22.
\end{aligned}$$

5.1. **Polar coordinates.** Let $g : (0,\infty)\times[0,2\pi) \to \mathbb{R}^2$ be the transformation $(\rho,\theta) \mapsto (x,y)$ be defined by

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta,$$

which is of class C^∞ and the Jacobian is

$$\det Jg(u,v) = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho\cos^2\theta + \rho\sin^2\theta = \rho(\cos^2\theta + \sin^2\theta) = \rho.$$

Let two sets A and D such that $A = g(D)$ and g is a bijection. Let $f : A \to \mathbb{R}$ such that the conditions of the Theorem of the change of variable are fulfilled. Then

$$\iint_A f(x,y)dxdy = \iint_D f(\rho\cos\theta, \rho\sin\theta)\rho d\rho d\theta.$$

Example 5.4. Calculate the volume V of the sphere of $\mathbb{R}^3$ of radius $r > 0$.

Solution. By symmetry, the volume V is twice the volume of the set

$$B = \{(x,y,z) : 0 \le z \le \sqrt{(r^2 - x^2 - y^2)}, x^2 + y^2 \le r^2\} \subseteq \mathbb{R}^3.$$

The volume of B is

$$\iint_A \sqrt{(r^2 - x^2 - y^2)}dxdy,$$

where $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le r^2\}$. Change to polar coordinates to see that the set A in polar coordinates is $D = \{(\rho,\theta) : 0 \le \rho \le r, 0 \le \theta < 2\pi\}$. Then

$$\begin{aligned}
V &= 2\iint_{[0,r]\times[0,2\pi]} \rho\sqrt{r^2 - \rho^2}d\rho d\theta \\
&= 2\int_0^r 2\pi\rho\sqrt{r^2 - \rho^2}d\rho \\
&= 2\pi\left(-\frac{2}{3}(r^2 - \rho^2)^{\frac{3}{2}}\Big|_0^r\right) = \frac{4}{3}\pi r^3.
\end{aligned}$$