April 6, 2022

III SEQUENCES AND SERIES

1. Mathematical Induction

To show that an statement P_n is true for any natural number n beginning with n_0, it is sufficient to prove that

(1) the statement is true for $n = n_0$;

(2) if the statement is true for some natural number $k \geq n_0$, then it is also true for the next natural number $k+1$.

This principle is known as the Principle of Mathematical Induction.

Example 1.1. Prove that $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$.

We shall use induction. If $n = 1$, then it is true. We assume that it holds true for a natural number k

$$1 + 2 + \cdots + k = \frac{1}{2}k(k+1).$$

adding $k+1$ to both side to this equality, we obtain

$$1 + 2 + \cdots + k + (k+1) = \frac{1}{2}k(k+1) + (k+1) = (k+1)\left(\frac{1}{2}k + 1\right) = \frac{1}{2}(k+1)(k+2).$$

We have proved that the equality holds true for $k+1$, hence it is valid for all $n \in \mathbb{N}$.

Example 1.2. Discover what is wrong in the following argument: "$\{1\}$ is a finite set, and, if $\{1, \ldots, n\}$ is a finite set, so is $\{1, \ldots, n+1\}$. Therefore the positive integers form a finite set."

2. Bounded and unbounded sequences

Definition 2.1. A sequence is an enumeration $x_1, x_2, \ldots, x_n, \ldots$ of real numbers. It is denoted by the symbol $\{x_n\}$ or (x_n), where the subindex $n = 1, 2, \ldots$. The number x_n is a term or element of the sequence and n is the number of the element.

Definition 2.2. Given two sequences $\{x_n\}$ and $\{y_n\}$, the sequences $\{x_n + y_n\}$, $\{x_n - y_n\}$, $\{x_n y_n\}$ and $\{x_n/y_n\}$ $(y_n \neq 0)$ are the sum, the difference, the product and the quotient of the sequences, respectively.

Definition 2.3. The sequence $\{x_n\}$ is bounded if there is $M > 0$ such that $|x_n| \leq M$ for all n.

Definition 2.4. It is said that sequence $\{x_n\}$ converges to $a \in \mathbb{R}$, and that a is the limit of the sequence $\{x_n\}$, written

$$\lim_{n\to\infty} x_n = a,$$

if and only if for all $\varepsilon > 0$, there is $N \geq 1$ such that for all $n > N$, $|x_n - a| < \varepsilon$.

Remark 2.5. This means that every interval with center at the point a, contains all terms of the sequence beginning with a certain number. A sequence with a limit is called a convergent sequence and a sequence that has not limit is called a divergent sequence.

Example 2.6. The sequence $\{1/n\}$ converges to 0. To prove this, let $\varepsilon > 0$. Since $\mathbb{N}$ is not bounded from above, there is $N \in \mathbb{N}$ such that $N > 1/\varepsilon$ (for instance, if ε is 10^{-6}, it suffices to take $N = 10^7$). Then $n > N$ implies $0 < 1/n < 1/N < \varepsilon$. This proves that the sequence $\{1/n\}$ converges to 0.

The sequence $\{\cos(n\pi)\}$ does not converge. Note that the sequence is in fact $\{-1, 1, -1, \ldots\}$. To prove this, let $\varepsilon = 1$. Let $N \in \mathbb{N}$. Then $|\cos((N+1)\pi) - \cos N\pi)| = 2 > \varepsilon$, thus the sequence is not convergent.

Proposition 2.7. *The limit of a convergent sequence is unique.*

Proposition 2.8 (Necessary condition of convergence)**.** *A convergent sequence is bounded.*

Definition 2.9. It is said that the sequence $\{x_n\}$ diverges to ∞ if and only if for all $M > 0$, there exists N such that for all $n > N$, $x_n > M$. It is said that the sequence $\{x_n\}$ diverges to $-\infty$ if and only if the sequence $\{-x_n\}$ diverges to ∞. It is designated

$$\lim_{n\to\infty} x_n = \infty, \qquad \text{and} \qquad \lim_{n\to\infty} x_n = -\infty,$$

respectively.

Example 2.10. The sequence $\{-n\}$ diverges to $-\infty$. To prove this, let $M < 0$ be arbitrary. Pick $N = M + 1$. Then for all $n > N$, $-n < -N < M$. Thus, the sequence $\{-n\}$ diverges to $-\infty$.

The sequence $\{(-1)^n n\}$ is not convergent and does not diverge neither to ∞ nor to $-\infty$.

3. Properties of convergent sequences

Theorem 3.1. *Let* $\lim_{n\to\infty} x_n = a$ *and* $\lim_{n\to\infty} y_n = b$. *Then*

(1) $\lim_{n\to\infty}(x_n \pm y_n) = a \pm b$.

(2) $\lim_{n\to\infty}(x_n y_n) = ab$.

(3) If $b \neq 0$, *then* $\lim_{n\to\infty}(x_n/y_n) = a/b$.

Remark 3.2. If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty}(x_n/y_n)$ is an indeterminate form $0/0$. If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$, then $\lim_{n\to\infty}(x_n - y_n)$ is an indeterminate form $\infty - \infty$. The indeterminate forms ∞/∞, $0 \cdot \infty$ are defined similarly.

Example 3.3. Calculate the limit of the sequence $\{x_n\}$, where $x_n = \sqrt{n^2+4n} - n$.

Note that the limit

$$\lim_{n\to\infty}(\sqrt{n^2+4n} - n)$$

is indeterminate, of the form $\infty - \infty$. We have

$$\begin{aligned}\lim_{n\to\infty}(\sqrt{n^2+4n} - n) &= \lim_{n\to\infty} \frac{(\sqrt{n^2+4n} - n)(\sqrt{n^2+4n} + n)}{\sqrt{n^2+4n} + n} \\ &= \lim_{n\to\infty} \frac{4n}{\sqrt{n^2+4n} + n} \\ &= \lim_{n\to\infty} \frac{4}{\sqrt{1+\frac{4}{n}} + 1} = 2.\end{aligned}$$

Theorem 3.4. *If* $\lim_{n\to\infty} x_n = a$ *and, beginning with a certain number* n, $x_n \geq b$, *then* $a \geq b$.

Theorem 3.5 (Three sequences theorem)**.** *If* $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = a$ *and, beginning with a certain number* n, *the inequalities* $x_n \leq z_n \leq y_n$ *hold true, then* $\lim_{n\to\infty} z_n = a$.

Example 3.6. Calculate

$$\lim_{n\to\infty} \frac{\sqrt{n}\sin n}{n^2+1}.$$

Let $z_n = \frac{\sqrt{n}\sin n}{n^2+1}$, $x_n = -\frac{\sqrt{n}}{n^2+1}$ and $y_n = \frac{\sqrt{n}}{n^2+1}$. Since $-1 \leq \sin n \leq 1$ for all $n \in \mathbb{N}$, we have

$$x_n \leq z_n \leq y_n,$$

for all $n \in \mathbb{N}$. Moreover,

$$\lim_{n\to\infty} -\frac{\sqrt{n}}{n^2+1} = -\lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1} = 0,$$

hence by the Three sequences Theorem, $\lim_{n\to\infty} \frac{\sqrt{n}\sin n}{n^2+1} = 0$.

4. Monotone sequences

Definition 4.1. A sequence $\{x_n\}$ is nonincreasing (nondecreasing) if and only if $x_{n+1} \leq x_n$ ($x_{n+1} \geq x_n$), for all n. A sequence $\{x_n\}$ is decreasing (increasing) if and only if $x_{n+1} < x_n$ ($x_{n+1} > x_n$), for all n.

Nonincreasing and nondecreasing sequences are known as monotone sequences.

Remark 4.2. A nonincreasing sequence is always bounded from above and a nondecreasing sequence is always bounded from below by the first term. If a monotone sequence is also bounded by the other side, then it is convergent.

Theorem 4.3. *A monotone bounded sequence converges.*

Example 4.4.

(1) Let the sequence $\{1/n\}$. The sequence is bounded, since $0 < 1/n < 1$ for all n, and it is decreasing, since $1/(n+1) < 1/n$, for all n. Hence, the sequence is convergent (as we already know, in fact this sequence converges to 0).

(2) Let the sequence $\{2n/(1+n)\}$. This sequence is bounded, since $0 < 2n/(1+n) < 2$ is equivalent to $0 < n$ and $2n < 2n+2$ which are obviously true. Moreover, the sequence is increasing, for

$$x_{n+1} - x_n = 2(n+1)/(2+n) - 2n/(1+n) = 2/((2+n)(1+n)) > 0.$$

Hence, the sequence is convergent (as we already know, in fact this sequence converges to 2).

(3) Let a sequence $\{x_n\}$ which is defined by the recurrence relation

$$x_{n+1} = x_n(2 - x_n), \quad n \geq 1, \tag{4.1}$$

where x_1 is an arbitrary number satisfying $0 < x_1 < 1$. Let us see that the sequence $\{x_n\}$ is bounded. We shall prove by induction that

$$0 < x_n < 1, \text{ for all } n \geq 1. \tag{4.2}$$

For $n = 1$ the inequality is satisfied. Suppose that the inequalities are true for the number n. We shall prove that then they are true for the number $n+1$. Note that the maximum of the function $x \mapsto x(2-x)$ in the interval $[0,1]$ is attained at $x = 1$, and the minimum is attained at 0. Hence, $0 < x_n < 1$ implies $0 < x_n(2 - x_n) < 1 \cdot (2 - 1) = 1$, Thus, $0 < x_{n+1} < 1$. We have thus proved inequalities (4.2). We shall now prove that the sequence is increasing. Since $x_n < 1$, $2 - x_n > 1$. Dividing (4.1) by x_n we have

$$\frac{x_{n+1}}{x_n} = 2 - x_n > 1.$$

Then, $x_{n+1} > x_n$, for all $n \geq 1$. Thus, the sequence $\{x_n\}$ is monotone and bounded. In consequence, it has a limit a. To find a, we pass to the limit in (4.1), to get

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n(2 - x_n), \quad \text{or} \quad a = a(2 - a).$$

Since $a = 0$ is not possible because $x_1 > 0$ and $\{x_n\}$ is increasing, we have $a = 1$.

Definition 4.5. Let $\{x_n\}$ be a sequence. Let $k_1 < k_2 < \cdots < k_n < \cdots$, be an arbitrary increasing sequence of positive integers (note that $k_n \geq n$). The sequence $\{x_{k_n}\}$, obtained from $\{x_n\}$ by choosing terms with numbers $k_1, k_2, \ldots, k_n, \ldots$, is called a subsequence of $\{x_n\}$.

Theorem 4.6. *If* $\lim_{\to\infty} x_n = a$, *then any subsequence* $\{x_{k_n}\}$ *converges to* a *as* $n \to \infty$.

5. Series

Definition 5.1. Let $\{a_n\}$ be a sequence. The formal expression

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$$

is called a series and the numbers a_k are the terms of the series. The number

$$S_n = \sum_{k=1}^{n} a_k$$

is the nth partial sum of the series.

Note that

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ &\vdots \\ S_n &= a_1 + a_2 + \cdots + a_n. \end{aligned}$$

Definition 5.2. It is said that the series $\sum_{k=1}^{\infty} a_k$ is convergent and its sum is S if and only if there is the limit

$$\lim_{n\to\infty} S_n = S.$$

If the limit does not exist or it is $\pm\infty$, we say that the series diverges.

Example 5.3. Let $q \neq 0$. The series

$$1 + q + q^2 + \cdots + q^n + \cdots$$

is the sum of geometric sequence with initial term 1 and ratio q. When $q \neq 1$

$$S_n = \frac{1-q^n}{1-q}.$$

If $|q| < 1$, then $|q|^n \to 0$ as $n \to \infty$, hence $q^n \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1}{1-q}.$$

Hence, when $|q| < 1$, $\sum_{k=1}^{\infty} q^{k-1} = \frac{1}{1-q}$.

If $|q| > 1$, then $|q^n| \to \infty$ as $n \to \infty$, hence $\frac{1-q^n}{1-q}$ converges to ∞ as $n \to \infty$ when $q > 1$, and has no limit when $q < 0$. Thus, the series geometric diverges when $|q| > 1$.

If $q = 1$, then $S_n = n$, $\lim_{n\to\infty} S_n = \infty$, and the series diverges.

If $q = -1$, then the series is alternate, with $S_n = 1$ if n is odd, and $S_n = -1$ if n is even, hence $\{S_n\}$ has not limit (since two subsequences have different limits), and the series diverges.

5.1. **Necessary condition of convergence.**

Theorem 5.4. *If a series $\sum_{k=1}^{\infty} a_k$ converges, then the sequence $\{a_n\}$ tends to 0.*

Proof. Let $S = \lim_{n\to\infty} S_n$. Then $S = \lim_{n\to\infty} S_{n-1}$. Also, $S_n - S_{n-1} = a_n$, hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

□

Corollary 5.5. *If $\{a_n\}$ does not converge to 0, then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

Example 5.6. The series

$$\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \cdots + \frac{n}{2n+1} + \cdots$$

diverges, since $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$.

Example 5.7. The harmonic series
$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$$
has general term that converges to 0 but the series diverges.

5.2. **Comparison of series of positive terms.** Let two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, with $a_k, b_k \geq 0$.

Theorem 5.8. *Suppose that $a_k \leq b_k$ for all $k = 1, 2, \ldots$.*

(1) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(2) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Example 5.9. Let the series
$$1+\frac{1}{2^2}+\frac{1}{3^3}+\cdots+\frac{1}{n^n}+\cdots.$$
Since $\frac{1}{n^n} \leq \frac{1}{2^n}$ for all $n \geq 2$ and that the series $\sum_{k^2}^{\infty} \frac{1}{2^k}$ is geometric of ratio $\frac{1}{2}$, the series above is convergent.

Example 5.10. Let the series
$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}+\cdots.$$
Since $\frac{1}{n} \geq \frac{1}{\sqrt{n}}$ for all $n \geq 1$ and that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, we have that the series above diverges.

5.3. **d'Alembert Criterion.**

Theorem 5.11. *Let a series $\sum_{k=1}^{\infty} a_k$ of positive terms such that the limit of the ratio of two consecutive terms is finite, that is,*
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < \infty.$$

(1) If $L < 1$, then the series converges.

(2) If $L > 1$, then the series diverges.

Example 5.12. Consider the series
$$\sum_{k=1}^{\infty} \frac{1}{k!}.$$
Since $\frac{n!}{(n+1)!} = \frac{1}{n+1}$ tends to 0 as $n \to \infty$, the series converges.

Example 5.13. Consider the series
$$\sum_{k=1}^{\infty} \frac{2^k}{k^2}.$$
Since $\frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = 2\frac{n^2}{(n+1)^2}$ tends to $2 > 1$ as $n \to \infty$, the series diverges.

5.4. **Cauchy Criterion.**

Theorem 5.14. *Let a series $\sum_{k=1}^{\infty} a_k$ of positive terms such that*
$$\lim_{n\to\infty} \sqrt[n]{a_n} = L < \infty.$$

(1) *If $L < 1$, then the series converges.*

(2) *If $L > 1$, then the series diverges.*

Example 5.15. Consider the series
$$\sum_{k=1}^{\infty} \left(\frac{k}{2k+1}\right)^k.$$
Since
$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1,$$
the series converges.

5.5. **Integral Criterion.**

Theorem 5.16. *Let a series $\sum_{k=1}^{\infty} a_k$ of positive terms, such that*
$$a_1 \geq a_2 \geq \cdots \geq a_k \geq a_{k+1} \geq \cdots,$$
and let $f : [1, \infty) \to [0, \infty)$ be a continuous and nonincreasing function such that
$$f(1) = a_1,\ f(2) = a_2, \ldots, f(n) = a_n, \ldots.$$

(1) *If the improper integral*
$$\int_1^{\infty} f(x)dx$$
converges, then the series converges.

(2) *If the improper integral*
$$\int_1^{\infty} f(x)dx$$
diverges, then the series diverges.

Example 5.17. Let the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}, \quad p > 0.$$

It is of positive terms and $\{\frac{1}{k^p}\}$ decreases, since $p > 0$. We can apply the integral criterion with the function $f(x) = \frac{1}{x^p}$. As we know, the improper integral

$$\int_1^{\infty} \frac{1}{x^p} dx$$

converges if and only if $p > 1$. Hence the series converges if and only $p > 1$.

5.6. **Alternate series.**

Definition 5.18. A series

$$a_1 - a_2 + a_3 - a_4 + \cdots ,$$

where $a_k > 0$, for all $k \geq 1$ is an alternate series.

Theorem 5.19 (Theorem of Leibniz)**.** *Let an alternate series*

$$a_1 - a_2 + a_3 - a_4 + \cdots ,$$

such that

$$a_1 > a_2 > \cdots > a_k > \cdots$$

and

$$\lim_{n \to \infty} a_n = 0.$$

Then the series converges, its sum S is positive, and $S < a_1$.

Remark 5.20. Let an alternate series satisfying the assumptions of Leibniz's Theorem,

$$a_1 - a_2 + a_3 - a_4 + \cdots .$$

Let $N \geq 1$ be an integer and consider the alternate series

$$a_{N+1} - a_{N+2} + \cdots .$$

By the theorem above, the series converges. If T_{N+1} denotes its sum, then $T_{N+1} > 0$ and $T_{N+1} < a_{N+1}$. Since

$$a_1 - a_2 + a_3 - a_4 + \cdots = S_N + (-1)^{N+2} \left(a_{N+1} - a_{N+2} + \cdots \right),$$

we have that

$$S - S_N = (-1)^{N+2} T_N.$$

If N is odd, then

$$0 > S - S_N = -T_N > -a_{N+1},$$

and if N is even, then

$$0 < S - S_N = T_N < a_{N+1}.$$

Hence the error in the approximation of the alternate series by truncating the series to N summands is less than a_{N+1}, that is,

$$|S - S_N| < a_{N+1}.$$

Moreover, if N is odd, the approximation is from above, and if N is even it is from below.

Example 5.21. The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is alternate, and convergent according to Leibniz's Theorem. Its sum, S, is positive and less than 1. What is the error if the correct sum of the series is estimated as

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = 0.7080\ldots?$$

Use $|S - S_5| < \frac{1}{5+1} = 0.1666$. Since $N = 5$ is odd, we get $0.7080 - 0.1666 = 0.5414 < S < 0.7080$.

5.7. **Series of positive and negative terms.**

Definition 5.22. The series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent iff $\sum_{k=1}^{\infty} |a_k|$ converges. If $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges, then the series $\sum_{k=1}^{\infty} a_k$ converges conditionally (or it is conditionally convergent).

Example 5.23. The series $\sum_{k=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.

The series $\sum_{k=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely.

Theorem 5.24. *If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then it converges.*

With finite sums, the order of the summands does not influence the result. We wonder whether this is also true with infinite sums.

Definition 5.25. Given the series $\sum_{k=1}^{\infty} a_k$ and a bijective mapping σ de $\{1, 2, \ldots\}$ to $\{1, 2, \ldots\}$, we say that the series $\sum_{k=1}^{\infty} a_{\sigma(k)}$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$.

Example 5.26. The mapping σ given by

$$\begin{aligned}1 &\mapsto 1\\ 2 &\mapsto 3\\ 3 &\mapsto 2\\ 4 &\mapsto 5\\ 5 &\mapsto 4\\ &\vdots\end{aligned}$$

rearranges the series $a_1 + a_2 + a_3 + \cdots$ into $a_1 + a_3 + a_2 + a_5 + a_4 + \cdots$. For instance, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$ transforms into $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \cdots$.

Note however that $a_1 + a_2 + a_4 + a_6 + \cdots$ is not a rearrangement, since some terms are missing not it is $a_1 + a_2 + a_2 + a_3 + a_3 + \cdots$, since some terms are repeated (in both case σ is not bijective).

Considering again the series $\sum_{k=1}^{\infty} a_k = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$, divide its term by 2 and intercalate zeroes, so we get a new series $\sum_{k=1}^{\infty} b_k$. If the initial series converges to S, then clearly the new series converges to $\frac{S}{2}$. Adding both series, we get another one which sum is $S + \frac{S}{2} = \frac{3S}{2}$. Here is a scheme of the operations:

$$\begin{aligned}\sum_{k=1}^{\infty} a_k &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots\\ \sum_{k=1}^{\infty} b_k &= 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 \cdots\\ \sum_{k=1}^{\infty} (a_k + b_k) &= 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \cdots\\ &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \cdots.\end{aligned}$$

The last line is a rearrangement of the intial series $1 - \frac{1}{2} + \frac{1}{3} - \cdots$, chich we have proved to converge to a different limit, $\frac{3S}{2}$.

Rearrangements have no effect when the series is absolutely convergent.

Theorem 5.27. *Let $\sum_{k=1}^{\infty} a_k$ an absolutely convergent series with sum S. Then the series $\sum_{k=1}^{\infty} a_{\sigma(k)}$ converges to S for any rearrangement σ.*