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II.3 IMPROPER INTEGRALS

1. Integration of bounded functions on unbounded intervals

Definition 1.1. Let $f : [a, +\infty) \to \mathbb{R}$ be integrable in each interval $[a, b]$, with $b > a$. We say that the improper integral of the first kind

$$\int_a^{+\infty} f(x)dx$$

converges if and only if the limit

$$\lim_{b\to\infty} \int_a^b f(x)dx$$

exists and is finite and in this case we define

$$\int_a^{+\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

Otherwise, we say that it diverges.

Definition 1.2. Let $f : (-\infty, b] \to \mathbb{R}$ be integrable in each interval $[a, b]$, with $a < b$. We say that the improper integral of the first kind

$$\int_{-\infty}^b f(x)dx$$

converges if and only if the limit

$$\lim_{a\to-\infty} \int_a^b f(x)dx$$

exists and is finite and in this case we define

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$

Otherwise, we say that it diverges.

Definition 1.3. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable in each interval $[a, b]$, with $a < b$. We say that the improper integral of the first kind

$$\int_{-\infty}^{+\infty} f(x)dx$$

converges if and only both

$$\int_{-\infty}^a f(x)dx \text{ and } \int_a^{+\infty} f(x)dx$$

converge for some a and in this case we define

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{+\infty} f(x)dx.$$

Otherwise, we say that it diverges.

Definition 1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable in each interval $[-a, a]$ for all a. The limit

$$\lim_{a\to\infty} \int_{-a}^{a} f(x)dx$$

is called the Cauchy principal value of $\int_{-\infty}^{+\infty} f(x)dx$.

Proposition 1.5. *If the improper integral $\int_{-\infty}^{+\infty} f(x)dx$ converges, then its value coincides with the Cauchy principal value, that is*

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{a\to\infty} \int_{-a}^{a} f(x)dx.$$

Note that the reciprocal of this proposition is not true. For instance the Cauchy principal value of $f(x) = x$ is 0, but $\int_{-\infty}^{+\infty} xdx$ diverges.

2. Integration of unbounded functions on bounded intervals

Definition 2.1. Let $f : (a, b] \to \mathbb{R}$ be a function that is not bounded in $(a, b]$ but is integrable in $[a + \varepsilon, b]$ for all $\varepsilon > 0$ such that $a + \varepsilon < b$. If the limit

$$\lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^{b} f(x)dx$$

exists and is finite, then we say that the improper integral of the second kind

$$\int_{a}^{b} f(x)dx$$

converges, and define

$$\int_{a}^{b} f(x)dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^{b} f(x)dx.$$

Otherwise, the integral is said to be divergent.

Definition 2.2. Let $f : [a, b) \to \mathbb{R}$ be a function that is not bounded in $[a, b)$ but is integrable in $[a, b - \varepsilon]$ for all $\varepsilon > 0$ such that $a < b - \varepsilon$. If the limit

$$\lim_{\varepsilon\to 0^+} \int_{a}^{b-\varepsilon} f(x)dx$$

exists and is finite, then we say that the improper integral of the second kind

$$\int_{a}^{b} f(x)dx$$

converges, and define

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)dx.$$

Otherwise, the integral is said to be divergent.

Definition 2.3. Let $f : [a, b] \to \mathbb{R}$ be a function and $c \in (a, b)$, such that f is not bounded, neither in $[a, c)$, nor in $(c, b]$, but is integrable in the intervals $[a, c - \varepsilon]$ and $[c + \varepsilon, b]$, for all $\varepsilon > 0$ such that $a < c - \varepsilon$ and $c - \varepsilon < b$. If the two improper integrals

$$\int_a^c f(x)dx \qquad \text{and} \qquad \int_c^b f(x)dx$$

converge, then we say that the improper integral of the second kind

$$\int_a^b f(x)dx$$

converges, and define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Otherwise, the integral is said to be divergent.

3. Integration of unbounded functions on unbounded intervals

Let $f : I \to \mathbb{R}$, where I is an unbounded interval ($I = (-\infty, b)$, $(-\infty, b]$, (a, ∞) or $[a, \infty)$), and f is not bounded in finitely many subintervals of I. Then, I can be partitioned in a finite number of intervals, $I_1, \ldots, I_n$, such that the integral

$$\int_{I_i} f(x)dx$$

is either an improper integral of the first kind, or an improper integral of the second kind.

We say that the integral

$$\int_I f(x)dx$$

converges if and only if each improper integral $\int_{I_i} f(x)dx$ converges. In this case, the improper integral of f on I is

$$\int_I f(x)dx = \int_{I_1} f(x)dx + \cdots + \int_{I_n} f(x)dx.$$

4. Absolute convergence of improper integrals

Definition 4.1. Let $[a,b)$, with $-\infty < a < b \leq +\infty$, and let $f:[a,b) \to \mathbb{R}$ be a function that is integrable in every interval $[a,x] \subseteq [a,b]$. We say that the improper integral $\int_a^b f(x)dx$ converges absolutely if and only if the improper integral

$$\int_a^b |f(x)|\,dx$$

converges.

Proposition 4.2. *If the improper integral $\int_a^b f(x)dx$ is absolutely convergent, then it is convergent and*

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx.$$

Remark 4.3. The reciprocal of this proposition is not true. For instance, the Dirichlet integral

$$\int_0^\infty \frac{\sin x}{x} dx$$

is convergent, but not absolutely.

5. Properties of the improper integral

Let $[a,b)$, with $-\infty < a < b \leq +\infty$.

(1) *Linearity.* If f and g are integrable in every interval $[a,x] \subseteq [a,b]$, and their improper integrals are convergent in $[a,b)$, then $\int_a^b(\alpha f(x)+\beta g(x))dx$ converges for all $\alpha, \beta \in \mathbb{R}$, and

$$\int_a^b (\alpha f(x) + \beta g(x))dx = (\alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

(2) Let $c \in (a,b)$ and let f be a function that is integrable in every interval $[a,x] \subseteq [a,b]$. Then the improper integral $\int_a^b f(x)dx$ converges if and only if $\int_c^b f(x)dx$ converges. In this case,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

(3) *Barrow's rule.* If $f:[a,b) \to \mathbb{R}$ is continuous, $F:[a,b) \to$ is an antiderivative of f, and $\int_a^b f(x)dx$ converges, then

$$\int_a^b f(x)dx = \lim_{x \to b^-} F(x) - F(a).$$

(4) *Change of variable.* Let $f : [a, b) \to \mathbb{R}$ be continuous and let $\phi : [\alpha, \beta) \to \mathbb{R}$ be of class C^1, where $-\infty < \alpha < \beta \leq +\infty$, $\phi(\alpha) = a$, $\lim_{t \to \beta^-} \phi(t) = b$ and the image $\phi([\alpha, \beta)) = [a, b)$ (the image of the interval $[\alpha, \beta)$ by ϕ is $[a, b)$). Then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt.$$

(5) *Integration by parts.* Let $u, v : [a, b) \to \mathbb{R}$ be two functions of class C^1 such that two of the following three integrals are convergent. Then the other integral will be convergent as well, and

$$\int_a^b u(x)v'(x)dx = \lim_{x \to b^-} (u(x)v(x)) - \int_a^b u'(x)v(x)dx.$$

6. Convergence criteria

Let $[a, b)$, with $-\infty < a < b \leq +\infty$.

(1) Let $f : [a, b) \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in [a, b)$, and f is integrable in very interval $[a, x] \subseteq [a, b)$. Then the improper integral $\int_a^b f(x)dx$ converges if and only if the function

$$F(x) = \int_a^x f(t)dt$$

is bounded in $[a, b)$.

(2) *Comparison criteria.* Let $f, g : [a, b) \to \mathbb{R}$ integrable in every $[a, x] \subseteq [a, b)$.

(a) Suppose that $0 \leq f(x) \leq g(x)$ for all $x \in [a, b)$. Then

(i) $\int_a^b g(x)dx$ converges $\Rightarrow$ $\int_a^b f(x)dx$ converges.

(ii) $\int_a^b f(x)dx$ diverges $\Rightarrow$ $\int_a^b g(x)dx$ diverges.

(b) Suppose that $f(x), g(x) > 0$ for all $x \in [a, b)$ and that

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = \ell.$$

Then

(i) $0 < \ell < +\infty \Rightarrow$ both $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ have the same character.

(ii) $\ell = 0$ and $\int_a^b g(x)dx$ converges $\Rightarrow$ $\int_a^b f(x)dx$ converges.

(iii) $\ell = +\infty$ and $\int_a^b g(x)dx$ diverges $\Rightarrow$ $\int_a^b f(x)dx$ diverges.

(c) Let $f : [a, b) \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in [a, b)$, and f is integrable in very interval $[a, x] \subseteq [a, b)$.

(i) Suppose that $b < +\infty$, that is, the interval $[a, b)$ is bounded, and

$$\lim_{x \to b^-} f(x)(b-x)^\alpha = k, \text{ with } -\infty < k < +\infty.$$

Then

$$\alpha < 1 \;\Rightarrow\; \int_a^b f(x)dx \text{ converges,}$$

$$\alpha \geq 1 \;\Rightarrow\; \int_a^b f(x)dx \text{ diverges}$$

(ii) Suppose that $b = +\infty$, that is, the interval is $[a, \infty)$, and

$$\lim_{x \to +\infty} f(x)x^\alpha = k, \text{ with } -\infty < k < +\infty.$$

Then

$$\alpha > 1 \;\Rightarrow\; \int_a^\infty f(x)dx \text{ converges,}$$

$$\alpha \leq 1 \;\Rightarrow\; \int_a^\infty f(x)dx \text{ diverges.}$$

Remark 6.1. All definitions and results above for improper integrals in the interval $[a, b)$, are equally valid for improper integrals in intervals $(a, b]$, with $-\infty \leq a < b < +\infty$.

7. Gamma and Beta functions

Definition 7.1 (Gamma function)**.** The gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx.$$

Remark 7.2. Note that Γ is an improper integral, which converges for all $p > 0$.

7.1. Properties of Γ.

(1) $\Gamma(1) = 1$.

(2) $\Gamma(p+1) = p\Gamma(p)$, for all $p > 0$.

(3) $\Gamma(n) = (n-1)!$, for all $p = 1, 2, \ldots$.

(4) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(5) $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}$, for all $0 < p < 1$.

Definition 7.3 (Beta function)**.** The beta function $\beta : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is defined as

$$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx.$$

Remark 7.4. Note that β is an improper integral, which converges for all $p > 0$, $q > 0$.

7.2. **Properties of β.**

(1) $\beta(p,q) = \beta(q,p)$, for all $p > 0$ and $q > 0$.

(2) $\beta(p,q) = 2\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} dx$, for all $p > 0$ and $q > 0$.

(3) $\beta(p,q) = \frac{\Gamma(p)}{\Gamma(q)}\Gamma(p+q)$, for all $p > 0$ and $q > 0$.

(4) $\beta(p,q) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$, for all $p, q = 1, 2, \ldots$.