April 27, 2022

# II.2 THE RIEMANN INTEGRAL

Integration is related to the calculus of areas under a curve. An intuitive way to find the area is to draw rectangles with smaller and smaller widths to obtain a good approximation. We hope than in the limit, that is, when the rectangles become infinitely thin, we get the exact value of the area. The Riemann integral formalizes this idea and establishes conditions for this approach to be successful.

## 1. Construction of the Riemann Integral

**Definition 1.1.** Let $A \subseteq \mathbb{R}$.

(1) $a$ is an upper bound of $A$ iff $a \geq x$, for all $x \in A$.

(2) $a$ is a lower bound of $A$ iff $a \leq x$, for all $x \in A$.

(3) $A$ is upper bounded iff $A$ has upper bounds.

(4) $A$ is lower bounded iff $A$ has lower bounds.

(5) $A$ is bounded iff it is both upper and lower bounded.

(6) $a$ is the supremum of $A$, written $a = \sup A$, iff $a$ is the least of the upper bounds of $A$, that is, $a \geq x$ for all $x \in A$ and $a \leq b$ for all upper bound $b$ of $A$.

(7) $a$ is the infimum of $A$, written $a = \inf A$, iff $a$ is the greatest of the lower bounds of $A$, that is, $a \leq x$ for all $x \in A$ and $a \geq b$ for all lower bound $b$ of $A$.

(8) $a$ is the maximum of $A$ iff $a = \sup A$ and $a \in A$.

(9) $a$ is the minimum of $A$ iff $a = \inf A$ and $a \in A$.

**Example 1.2.** $A = (0, \infty)$ is lower bounded but not upper bounded; $\inf A = 0$; $A$ has no minimum, since $0 = \inf A \notin A$.

$B = (-\infty, 0)$ is upper bounded but not lower bounded; $\sup B = 0$; $B$ has no maximum, since $0 = \sup B \notin B$.

**Definition 1.3.** A partition of the interval $[a, b]$ is a finite set of points

$$\mathcal{P} = \{x_0, x_1, \dots, x_n\},$$

where $a = x_0 < x_1 < \cdots < x_n = b$.

**Definition 1.4.** Given two partitions $\mathcal{P}$ and $\mathcal{P}'$ of the interval $[a,b]$, we say that $\mathcal{P}'$ is finer than $\mathcal{P}$ if and only if $\mathcal{P} \subseteq \mathcal{P}'$.

**Definition 1.5.** Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let $\mathcal{P}$ be a partition of $[a,b]$ and let

$$m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\},$$
$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\},$$

for all $i = 1, \ldots, n$.

The lower Darboux sum of $f$ on $\mathcal{P}$ is

$$s(f, \mathcal{P}) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}).$$

The upper Darboux sum of $f$ on $\mathcal{P}$ is

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}).$$

**Example 1.6.** Compute the lower and upper Darboux sums of $f(x) = x^2$ in the interval $[-3,3]$ when the partitions are $\mathcal{P} = \{-3,-2,0,2,3\}$ and $\mathcal{P}' = \{-3,-2,-1,0,1,2,3\}$.

It is easy to calculate $m_1 = f(-2) = 4$, $m_2 = m_3 = f(0) = 0$, $m_4 = f(2) = 4$, and $M_1 = f(-3) = 9$, $M_2 = f(-2) = 4$, $M_3 = f(2) = 4$, $M_4 = f(3) = 9$. Hence

$$s(x^2, \mathcal{P}) = 4(-2-(-3)) + 0(0-(-2)) + 0(2-0) + 4(3-2) = 8,$$
$$S(x^2, \mathcal{P}) = 9(-2-(-3)) + 4(0-(-2)) + 4(2-0) + 9(3-2) = 34.$$

On the other hand, $m_1' = f(-2) = 4$, $m_2' = f(-1) = 1$, $m_3' = m_4' = f(0) = 0$, $m_5' = f(1) = 1$, $m_6' = f(2) = 4$, and $M_1' = f(-3) = 9$, $M_2' = f(-2) = 4$, $M_3' = f(-1) = 1$, $M_4' = f(1) = 1$, $M_5' = f(2) = 4$, $M_6' = f(3) = 9$. Hence (noting that the increment $x_i - x_{i-1} = 1$)

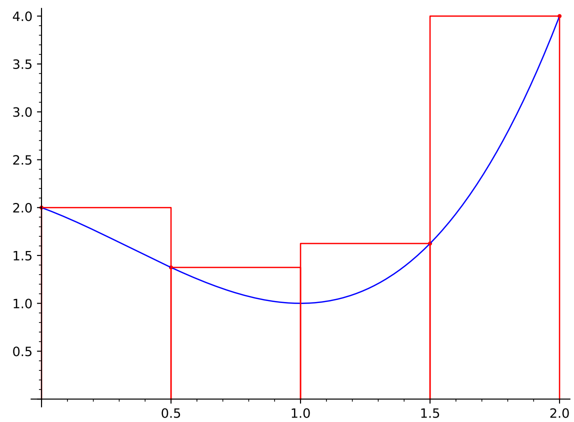
$$s(x^2, \mathcal{P}') = 4 + 1 + 0 + 0 + 1 + 4 = 10,$$
$$S(x^2, \mathcal{P}') = 9 + 4 + 1 + 1 + 4 + 9 = 28.$$

Note that $\mathcal{P}'$ is finer than $\mathcal{P}$ and

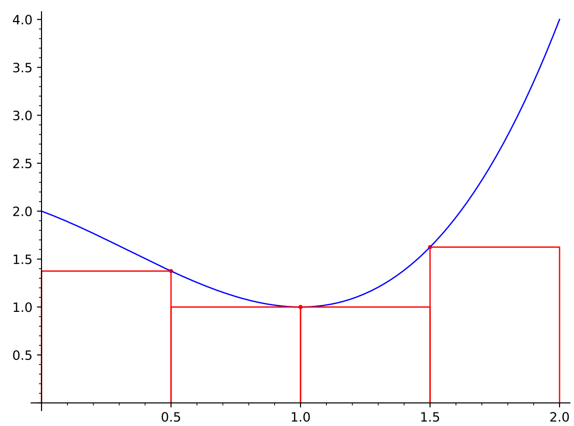
$$s(x^2, \mathcal{P}) \le s(x^2, \mathcal{P}') \le S(x^2, \mathcal{P}') \le S(x^2, \mathcal{P}).$$

**Example 1.7.** Let the function $f(x) = x^3 - x^2 - x + 2$ and the interval $[0, 2]$. The figures below illustrate the upper and the lower Darboux sums for partitions $\{0, 0.5, 1, 1.5, 2\}$ and $\{0, 0.25, 0.5, 0.75, 1.25, 1.5, 1.75, 2\}$, respectively. Note how the approximation is better as the partition is finer. The true value of the integral is

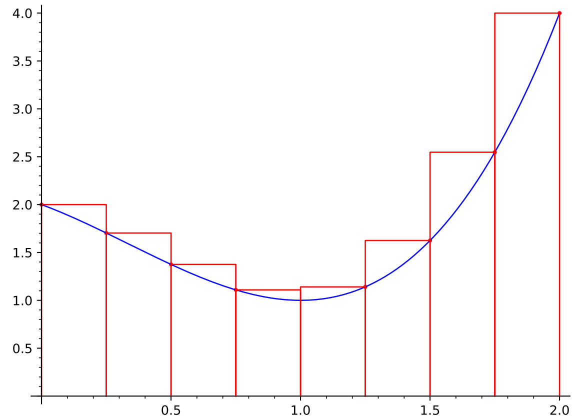
$$\int_0^2 f(x)\, dx = \frac{10}{3} \approx 3.33333,$$



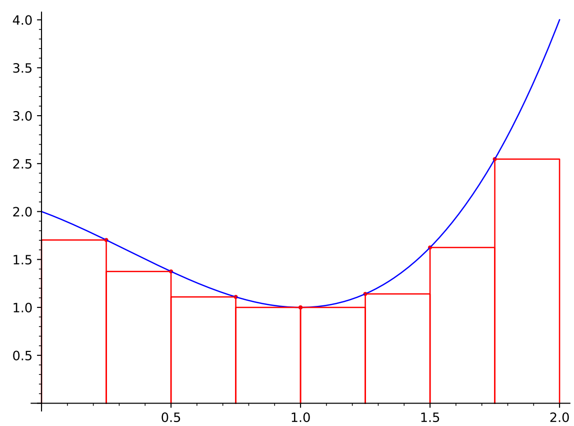

$$\begin{aligned}\sum_{i=1}^{4} M_i\, \Delta x &= 1/2 \cdot [f(0) + f(0.50) + f(1.50) + f(2.00)] \\ &= 1/2 \cdot [2.0000 + 1.3750 + 1.6250 + 4.0000] \\ &= 4.50.\end{aligned}$$




$$\begin{aligned}\sum_{i=1}^{4} m_i\, \Delta x &= 1/2 \cdot [f(0.50) + f(1.00) + f(1.00) + f(1.50)] \\ &= 1/2 \cdot [1.3750 + 1.0000 + 1.0000 + 1.6250] \\ &= 2.50.\end{aligned}$$




$$\begin{aligned}\sum_{i=1}^{8} M_i\,\Delta x &= 1/4\cdot[f(0)+f(0.25)+f(0.50)+f(0.75)+f(1.25)+f(1.50)+f(1.75)+f(2.00)]\\ &= 1/4\cdot[2.0000+1.7031+1.3750+1.1094+1.1406+1.6250+2.5469+4.0000]\\ &= 3.8750.\end{aligned}$$




$$\begin{aligned}\sum_{i=1}^{8} m_i\,\Delta x &= 1/4\cdot[f(0.25)+f(0.50)+f(0.75)+f(1.00)+f(1.00)+f(1.25)+f(1.50)+f(1.75)]\\ &= 1/4\cdot[1.7031+1.3750+1.1094+1.0000+1.0000+1.1406+1.6250+2.5469]\\ &= 2.8750.\end{aligned}$$

**Proposition 1.8** (Properties of Darboux sums)**.** *Let $f:[a,b]\to\mathbb{R}$ be bounded, with*

$$m=\inf\{f(x)\,:\,x\in[a,b]\},$$
$$M=\sup\{f(x)\,:\,x\in[a,b]\}.$$

*Let $\mathcal{P}$, $\mathcal{P}'$ two partitions of* $[a,b]$.

*(1)* $m(b-a) \leq s(f, \mathcal{P}) \leq S(f, \mathcal{P}) \leq M(b-a)$.

*(2) If* $\mathcal{P}'$ *is finer than* $\mathcal{P}$*, then*

$$s(f, \mathcal{P}) \leq s(f, \mathcal{P}') \leq S(f, \mathcal{P}') \leq S(f, \mathcal{P}).$$

*(3)* $s(f, \mathcal{P}') \leq S(f, \mathcal{P})$.

*(4) The sets*

$$\{s(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\} \text{ and } \{S(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}$$

*are bounded.*

Property (4) is immediate from properties (1) and (3). For instance, $0 \leq s(x^2, \mathcal{P}) \leq S(x^2, \mathcal{P}) \leq 54$, for all partition $\mathcal{P}$. In fact, using property (2), we can give a better estimate for the area under the graph of $f(x) = x^2$ in the interval $[-3, 3]$ (assuming that the area exists): $10 \leq \text{area} \leq 28$.

**Definition 1.9.** Let $f : [a, b] \to \mathbb{R}$ be bounded.

The lower integral of $f$ in $[a, b]$ is defined as the number

$$L \int_a^b f = \sup\{s(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}.$$

The upper integral of $f$ in $[a, b]$ is defined as the number

$$U \int_a^b f = \inf\{S(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}.$$

**Proposition 1.10.** *Let* $f : [a, b] \to \mathbb{R}$ *be bounded. Then*

$$L \int_a^b f \leq U \int_a^b f.$$

**Definition 1.11.** Let $f : [a, b] \to \mathbb{R}$ be bounded.

We say that the function is Riemann integrable (or simply integrable) in the interval $[a, b]$ iff

$$L \int_a^b f = U \int_a^b f.$$

In this case, this number is the integral of $f$ in $[a, b]$ (or defined integral of $f$ in $[a, b]$) and is denoted

$$\int_a^b f, \quad \int_a^b f(x) dx.$$

When $f$ is nonnegative in $[a,b]$, $\displaystyle\int_a^b f(x)dx$ is the area of the region of the plane

$$\{(x,y)\,:\, a \le x \le b,\ 0 \le y \le f(x)\}.$$

Not every bounded function is integrable.

**Theorem 1.12.** *The function $f : [a,b] \to \mathbb{R}$ is integrable iff for all $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that*

$$S(f,\mathcal{P}) - s(f,\mathcal{P}) < \varepsilon.$$

**Proposition 1.13.** *If $f$ is continuous in $[a,b]$, then $f$ is integrable in $[a,b]$.*

This result admits a useful generalization.

**Proposition 1.14.** *If $f$ is bounded in $[a,b]$ and has a finite number of discontinuities, then $f$ is integrable in $[a,b]$.*

**Example 1.15.** The only point of discontinuity of the signum function $\operatorname{sgn}(x) = 1$, if $x > 1$, $\operatorname{sgn}(x) = -1$, if $x < 0$, is 0, hence sgn is an integrable function. Also, the value of the integral does not change by attaching any value to $\operatorname{sgn}(0)$. To compute $\int_{-2}^{2} \operatorname{sgn}(x)dx$, we can decompose the integral into the sum $\int_{-2}^{0}(-1)dx + \int_0^2 1dx$. Using the Barrow's rule in each interval $[-2,0]$ and $[0,2]$ (see Theorem 3.2 below), we have

$$\int_{-2}^{2} \operatorname{sgn}(x)dx = -x\Big|_{-2}^{0} + x\Big|_{0}^{2} = 0.$$

Hence the signum function is integrable in $[-2,2]$ and its integral is 0.

## 2. Properties of the Riemann Integral

**Proposition 2.1** (Properties of the integral)**.** Let $f,g : [a,b] \to \mathbb{R}$ be integrables and let $\alpha \in \mathbb{R}$.

(1) *Linearity.*

  (a) $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

  (b) $\int_a^b \alpha f = \alpha \int_a^b f$.

(2) *Monotonicity.*

$$f(x) \ge g(x) \text{ for all } x \in [a,b], \text{ implies } \int_a^b f \ge \int_a^b g.$$

(3) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$

(4) *Additivity with respect to the interval.* $f$ is integrable if and only if $f$ is integrable in $[a,c]$ and in $[c,b]$ for all $c \in [a,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

**Proposition 2.2** (Theorem of the mean)**.** *Let $f : [a,b] \to \mathbb{R}$ be integrable.*

*(1) If $f$ is integrable in $[a,b]$ and if $m$ and $M$ are lower and upper bounds of $f$ in $[a,b]$, respectively (they could be the infimum and the supremum), then there is $\alpha \in [m,M]$ such that*

$$\int_a^b f(x) = \alpha(b-a).$$

*(2) If $f$ is continuous in $[a,b]$, then there exists $c \in [a,b]$ such that*

$$\int_a^b f(x) = f(c)(b-a).$$

## 3. Fundamental Theorem of Calculus

In this section we show the connection between Riemann integral and antiderivatives.

**Theorem 3.1** (Fundamental Theorem of Calculus)**.** *Let $f : [a,b] \to \mathbb{R}$ be integrable and let $F : [a,b] \to \mathbb{R}$ be defined by*

$$F(x) = \int_a^x f(t)\,dt. \tag{3.1}$$

*Then*

*(1) $F$ is continuous in $[a,b]$.*

*(2) If $f$ is continuous in $x \in [a,b]$, then $F$ is derivable in $x$ and $F'(x) = f(x)$ (hence, if $f$ is continuous in $[a,b]$, then $F$ is an antiderivative of $f$).*

Written in other terms, the theorem establishes

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

**Theorem 3.2** (Barrow's Rule)**.** *Let $f : [a,b] \to \mathbb{R}$ be integrable and let $G$ be an antiderivative of $f$ in $[a,b]$ (that is, $G'(x) = f(x)$ for all $x \in [a,b]$). Then*

$$\int_a^b f(x)\,dx = G(b) - G(a),$$

*Proof.* Since that both $G$ and the function $F$ defined in (3.1) are antiderivatives of $f$ in $[a, b]$, the difference $G - F$ is constant in $[a, b]$. Hence $G(a) - F(a) = G(b) - F(b)$, or

$$G(b) - G(a) = F(b) - F(a) = \int_a^b f(x)\,dx - \int_a^a f(x)\,dx = \int_a^b f(x)\,dx.$$

□

Most often we will write $G(b) - G(a)$ as $G(x)\Big|_a^b$.

**Theorem 3.3** (Change of variable). *Let $f$ be continuous in $[a, b]$, and let $x = g(t)$ be continuous, together with the derivative in $[\alpha, \beta]$, where $g(\alpha) = a$, $g(\beta) = b$ and $a \leq g(t) \leq b$. Then*

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))g'(t)\,dt.$$

Note: from $x = g(t)$ one gets $dx = g'(t)dt$, and the identity above follows.

**Theorem 3.4** (Integration by parts). *If $u$ and $v$ have continuous derivatives in $[a, b]$, then*

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx.$$

## 4. The area of a plane figure

Given a continuous function $f$, the area of the figure bounded by the curve $y = f(x)$, the axis $OX$ and the line segments $x = a$, $x = b$ is

$$A = \int_a^b |f(x)|\,dx.$$

**Example 4.1.** The area of the figure limited by $y = 1 - x$ in the interval $[0, 2]$ is

$$A = \int_0^2 |1 - x|\,dx = \int_0^1 (1 - x)\,dx + \int_1^2 -(1 - x)\,dx = \frac{1}{2} + \frac{1}{2} = 1.$$

**Example 4.2.** The area of the figure limited by the graph of $y = \ln x$ and the horizontal axis and the line segments $x = 1/e$, $x = e$ is

$$A = \int_{\frac{1}{e}}^e |\ln x|\,dx.$$

The logarithm is negative in $[1/e, 1]$ and positive in $[1, e]$. thus

$$A = \int_{\frac{1}{e}}^1 -\ln x\,dx + \int_1^e \ln x\,dx$$

The integral can be solved using parts $u = \ln x$, $dv = dx$ obtaining

$$\int_{\frac{1}{e}}^{1} \ln x \, dx = x \ln x \Big|_{\frac{1}{e}}^{1} - x \Big|_{\frac{1}{e}}^{1} = -1 + \frac{2}{e},$$

$$\int_{1}^{e} \ln x \, dx = x \ln x \Big|_{1}^{e} - x \Big|_{1}^{e} = 1.$$

Thus, $A = -(-1 + 2/e) + 1 = 2(1 - 1/e)$.

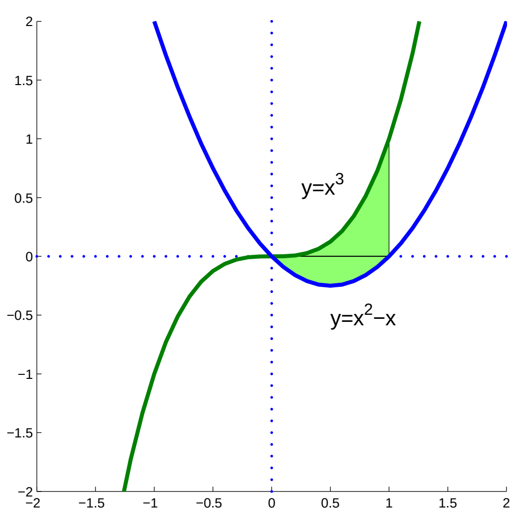
Suppose that a plane figure is bounded by the continuous curves $y = f(x)$, $y = g(x)$, $a \le x \le b$, where $g(x) \le f(x)$, and two line segments $x = a$, $x = b$ (the line segments may degenerate into a point). Then the area of the figure is

$$A = \int_a^b (f(x) - g(x))\, dx.$$

**Example 4.3.** Find the area of the figure bounded by the curves $y = x^3$, $y = x^2 - x$ in the interval $[0, 1]$.

Solution: The curves meet at a single point. Solving the equation $x^3 = x^2 - x$, we find the abscissa of the point, $x = 0$. Hence one of the curves remains above the other in the whole interval. To know which, we simply substitute into $x^3 - x^2 + x$ an arbitrary value in the interval; for $x = 1/2$ we get $x^3 - x^2 + x|_{x=1/2} = 0.375 > 0$, thus $x^3$ is above $x^2 - x$ in $[0, 1]$. The area is

$$A = \int_0^1 x^3 - (x^2 - x)\, dx = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2}\Big|_0^1 = \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{2}\right) - 0 = \frac{5}{12}.$$



**Example 4.4.** Find the area of the figure bounded by the curves $y = 2 - x^2$, $y = x$.

SOLUTION: The curves meet at two points. Solving the equation $2 - x^2 = x$ we find that the points are $x = -2$, $x = 1$. Hence one of the curves remains above the other in the interval $[-2, 1]$. To know which, we simply substitute into $2 - x^2 = x$ an arbitrary value on the interval $[-2, 1]$; for $x = 0$, $2 - x^2 - x|_{x=0} = 2 > 0$, thus $y = 2 - x^2$ is above $Y = x$ in $[-2, 1]$. The area is

$$A = \int_{-2}^{1} 2 - x^2 - x\, dx = 2x - \frac{x^3}{3} - \frac{x^2}{2}\Big|_{-2}^{1} = \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right) = \frac{9}{2}.$$