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II.1 CALCULUS OF PRIMITIVES

1. Antiderivatives. The indefinite integral

The problem we set in this lesson is the following: given a function f, is there a function F such that $F' = f$?

Definition 1.1. We say that the function $F : [a,b] \to \mathbb{R}$ is differentiable in the interval $[a,b]$, if F is differentiable in (a,b) and the two following limits exist and are finite

$$F'(a^+) := \lim_{h\to 0^+} \frac{F(a+h)-F(a)}{h} \quad \text{and} \quad F'(b^-) := \lim_{h\to 0^-} \frac{F(a+h)-F(a)}{h}.$$

We say that F is of class C^1 in $[a,b]$ if the derivative of order 1 of F is continuous in $[a,b]$.

In the above, we could replace $[a,b]$ by $[a,b)$ or $(a,b]$.

Example 1.2. The function $F(x) = x$ is differentiable in $[0,1]$, but $G(x) = \sqrt{x}$ is not, since $G'(0^+) = \infty$.

In what follows, I denotes an interval $[a,b]$.

Definition 1.3. The function F is an antiderivative (or primitive) of the function f on the interval I if and only if $F'(x) = f(x)$ for all $x \in I$.

Example 1.4. Both $F_1(x) = x^3 + 6$ and $F_2(x) = x^3 - 2$ are antiderivatives of $f(x) = 3x^2$ in any interval $[a,b]$.

Theorem 1.5. *If F_1 and F_2 are two arbitrary antiderivatives of f on I, then $F_1(x) - F_2(x) =$ const. on I.*

Proof. By definition of antiderivative, $F_1' = F_2' = f$ on I, thus $(F_1 - F_2)'(x) = 0$ for every $x \in I$. Since a function with a null derivative on an interval is constant, we have $F_1(x) - F_2(x) = \text{const}$. □

Corollary 1.6. *If F is one of the antiderivatives of f on I, then any other antiderivative G of the function f on I has the form $G(x) = F(x) + C$, where C is a constant.*

Definition 1.7. The set of all antiderivatives of the function f on the interval I is called the indefinite integral of f on I, and it is denoted

$$\int f(x)\,dx.$$

Remark 1.8. Note that by Corollary 1.6, $\int f(x)\,dx = F(x) + C$, where F is one of the antiderivatives of f on I, and C is an arbitrary constant.

Example 1.9.
(1) An antiderivative of the constant function $f(x) = 1$ is x, thus $\int 1dx = x + C$. In the same way, $\int(-1)dx = -x + C$
(2) Since $(\sin x)' = \cos x$, we have $\int \cos x dx = \sin x + C$. The antiderivative of $f(x) = \cos x$ that takes the value -5 at the point $x = \pi/2$ is $F(x) = \sin x - 6$ (this comes from the equality $\sin \pi/2 + C = -5$).
(3) Not every function has an antiderivative in a given interval. Let the function

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0; \\ 0, & \text{if } x = 0; \\ 1, & \text{if } x > 0; \end{cases}$$

and the interval $I = [-2, 2]$. If F is an antiderivative of sgn in this interval, then $F'(x) = -1$ for $x \in [-2, 0)$ and $F'(x) = 1$ for $x \in (0, 2]$. Thus, $F(x) = -x + C_1$ for $x \in [-2, 0)$, and $F(x) = x + C_2$ for $x \in (0, 2]$. Since that F has to be continuous in the interval $[-2, 2]$, it has to be continuous at 0, thus $F(0) = C_1 = C_2$. Hence, $F(x) = |x| + C$ for all $x \in [-2, 2]$. But F is not differentiable at 0, so we obtain a contradiction.

1.1. **Properties of the Indefinite Integral.**

(1) $\int F'(x)\,dx = F(x) + C$;
(2) For any functions f, g and constants α, β,

$$\int (\alpha f(x) + \beta g(x))\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx.$$

1.2. **Basic Indefinite Integrals.**

(1) $\displaystyle\int 0\,dx = C$;
(2) $\displaystyle\int 1\,dx = x + C$;

(3) $\displaystyle\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \qquad (a \neq -1);$

(4) $\displaystyle\int \frac{dx}{x} = \ln|x| + C \qquad (x \neq 0);$

(5) $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (0 < a \neq 1),\ \int e^x\,dx = e^x + C;$

(6) $\displaystyle\int \sin x\,dx = -\cos x + C;$

(7) $\displaystyle\int \cos x\,dx = \sin x + C;$

(8) $\displaystyle\int \frac{1}{\cos^2 x}\,dx = \tan x + C \qquad (x \neq \frac{\pi}{2} + k\pi,\ k \text{ integer});$

(9) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \qquad (-1 < x < 1);$

(10) $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C.$

2. Integration by Change of Variable

Sometimes the task of finding the integral $\displaystyle\int f(x)\,dx$ is simplified by means of a change of variable $x = \varphi(t)$.

Theorem 2.1. *Let I, J two intervals of $\mathbb{R}$ and let $\varphi : J \to \mathbb{R}$ be of class C^1 in J, such that $\varphi(J) \subseteq I$. Let $f : I \to \mathbb{R}$ be continuous and suppose that F is an antiderivative of f on the interval I. Then the function $F(\varphi(t))$ is an antiderivative of the function $f(\varphi(t))\varphi'(t)$ and hence*

$$\int f(\varphi(t))\varphi'(t)dt = F(\varphi(t)) + C.$$

Remark 2.2. It follows form Theorem 2.1 that

$$\int f(x)dx\Big|_{x=\varphi(t)} = \int f(\varphi(t))\varphi'(t)dt.$$

This is the formula of change of variable. If the function $x = \varphi(t)$ admits and inverse $t = \varphi^{-1}(x)$, then the formula can be rewritten

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt\Big|_{t=\varphi^{-1}(x)},$$

which is the formula generally used to evaluate the integral $\int f(x)dx$ by the method of change of variable.

Example 2.3. Find $\displaystyle\int \tan x \, dx$.

Solution: Let $t = \cos x$. Then $dt = -\sin x \, dx$. Hence, by the formula of change of variable

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{dt}{t} = -\ln|t| + C = -\ln|\cos x| + C.$$

Example 2.4. Find $\displaystyle\int \sqrt{2x-1} \, dx$.

Solution: Let $t = 2x - 1$. Then $dt = 2dx$. Hence,

$$\int \sqrt{2x-1} \, dx = \frac{1}{2} \int \sqrt{t} \, dt = \frac{1}{2} \int t^{1/2} \, dt = \frac{1}{2} \frac{t^{3/2}}{3/2} + C = \frac{1}{3} t^{3/2} + C = \frac{1}{3}(2x-1)^{3/2} + C.$$

Example 2.5. Find $\displaystyle\int x\sqrt{2x-1} \, dx$.

Solution: Let $t = 2x - 1$. Then $dt = 2dx$. Moreover, $x = (1+t)/2$. Applying the change of variable formula we get

$$\begin{aligned}\int x\sqrt{2x-1} \, dx &= \frac{1}{4} \int (1+t)t^{1/2} \, dt = \frac{1}{4} \int t^{1/2} + t^{3/2} \, dt = \frac{1}{4}\left(\frac{t^{3/2}}{3/2} + \frac{t^{5/2}}{5/2}\right) + C \\ &= \frac{3}{2}(2x-1)^{3/2} + \frac{5}{2}(2x-1)^{3/2} + C.\end{aligned}$$

Example 2.6. Find $\displaystyle\int \frac{\ln x}{x} \, dx$.

Solution: Let $t = \ln x$. Then $dt = dx/x$ and

$$\int \frac{\ln x}{x} \, dx = \int t \, dt = \frac{1}{2} t^2 + C = \frac{1}{2}(\ln x)^2 + C.$$

Example 2.7. Find $\displaystyle\int x e^{-x^2} \, dx$.

Solution: Let $t = x^2$. Then $dt = 2xdx$ and

$$\int x e^{-x^2} \, dx = \frac{1}{2} \int e^{-t} \, dt = -\frac{1}{2} e^{-t} + C = -\frac{1}{2} e^{-x^2} + C.$$

3. Integration by parts

For differentiable functions u and v we have $(uv)' = uv' + vu'$. Taking integrals and given that $\int (uv)'(x)\, dx = u(x)v(x)$, we have the following formula of integration by parts.

Theorem 3.1. *Let $u, v : I \to \mathbb{R}$ be of class C^1 in the interval I. Then*

$$\int u(x)v'(x)\, dx = u(x)v(x) - \int v(x)u'(x)\, dx.$$

Remark 3.2. This relation is known as the *formula of integration by parts.* Using the identifications $u'(x)\, dx = du$ and $v'(x)\, dx = dv$ we can write this formula as

$$\int u\, dv = uv - \int v\, du.$$

Example 3.3. Find $\displaystyle\int xe^x\, dx$.

Solution: Let $u = x$ and $dv = e^x\, dx$. Then $du = dx$ and $v = e^x$. Hence

$$\int xe^x\, dx = xe^x - \int e^x\, dx = e^x(x-1) + C.$$

Example 3.4. Find $\displaystyle\int x^2 \ln x\, dx$.

Solution: Let $u = \ln x$ and $dv = x^2\, dx$. Notice that $du = dx/x$ and $v = x^3/3$. Then, using the formula of integration by parts we get

$$\int x^2 \ln x\, dx = \ln x \left(\frac{x^3}{3}\right) - \int \frac{x^3}{3x}\, dx = \ln x \left(\frac{x^3}{3}\right) - \frac{1}{3}\int x^2\, dx = \ln x \left(\frac{x^3}{3}\right) - \frac{1}{9}x^3 + C.$$

Example 3.5. Find $\displaystyle\int \arctan x\, dx$.

Solution: Let $u = \arctan x$ and $dv = dx$. Then $du = dx/(1+x^2)$ and $v = x$. Hence

$$\int \arctan x\, dx = x \arctan x - \int \frac{x}{1+x^2}\, dx.$$

Now, observe that using the change of variable $t = x^2$ we have $dt = 2x\, dx$ thereby

$$\int \frac{x}{1+x^2}\, dx = \frac{1}{2}\int \frac{1}{1+t}\, dt = \frac{1}{2}\ln|1+t| + C = \frac{1}{2}\ln(1+x^2) + C.$$

Plugging this value into the above expression we finally get

$$\int \arctan x\, dx = x \arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

Example 3.6. Find $\int x^2 \sin x \, dx$.

Solution: Let $u = x^2$ and $dv = \sin x \, dx$. Then $du = 2x \, dx$ and $v = -\cos x$. Thus

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx.$$

Applying again integration by parts to the second integral, $u = x$ and $dv = \cos x \, dx$ we have $du = dx$ and $v = \sin x$, hence

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

Plugging this value into the previous expression we finally get

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

4. Integration of Rational Functions

A rational function is of the form $\frac{P_n(x)}{Q_m(x)}$, where P_n and Q_m are polynomials of degrees n and m, respectively.

4.1. **Decomposition into simple fractions.** If $n \geq m$ the fraction is *improper* and can be represented

$$\frac{P_n(x)}{Q_m(x)} = P_{n-m}(x) + \frac{R_k(x)}{Q_m(x)},$$

where the degree of the polynomial R_k is $k < m$. Thus, the integration of an improper fraction can be reduced to the integration of a *proper* fraction ($k < m$) and a polynomial.

$$\int \frac{P_n(x)}{Q_m(x)} \, dx = \int P_{n-m}(x) \, dx + \int \frac{R_k(x)}{Q_m(x)} \, dx.$$

Example 4.1.

$$\int \frac{x^3 + x^2 + x}{x^2 + 1} \, dx = \int (x+1) \, dx - \int \frac{1}{x^2+1} \, dx,$$

because dividing the polynomials we find

$$\frac{x^3 + x^2 + x}{x^2 + 1} = x + 1 - \frac{1}{x^2+1}.$$

Then

$$\int \frac{x^3 + x^2 + x}{x^2 + 1} \, dx = \frac{1}{2}(x+1)^2 - \arctan x + C.$$

When $n < m$, we will use the following theorem. It asserts that any proper fraction can be decomposed into simple fractions. Although we establish the theorem with full generality, only the simplest cases will be worked on in the course.

Theorem 4.2. *Suppose that* $\dfrac{P_n(x)}{Q_m(x)}$ *is an proper fraction ($n < m$) and that*

$$Q_m(x) = (x-a)^\alpha \cdots (x-b)^\beta ((x-p)^2+q^2)^\gamma \cdots ((x-r)^2+s^2)^\delta,$$

where $a, \ldots, b$ are real roots of multiplicity $\alpha, \ldots, \beta$ and $(x-p)^2+q^2, \ldots, (x-r)^2+s^2$ are quadratic trinomials irreducible into real factors of multiplicity $\gamma, \ldots, \delta$ (notice that $\alpha, \ldots, \beta, \gamma, \ldots, \delta$ are all positive integers). Then there are constants A_i, B_i, M_i, N_i, K_i and L_i such that

$$\begin{aligned}\frac{P_n(x)}{Q_m(x)} &= \frac{A_\alpha}{(x-a)^\alpha} + \frac{A_{\alpha-1}}{(x-a)^{\alpha-1}} + \cdots + \frac{A_1}{x-a}\\ &\quad + \cdots + \frac{B_\beta}{(x-b)^\beta} + \frac{B_{\beta-1}}{(x-b)^{\beta-1}} + \cdots + \frac{B_1}{x-b}\\ &\quad + \frac{M_\gamma x + N_\gamma}{((x-p)^2+q^2)^\gamma} + \frac{M_{\gamma-1}x + N_{\gamma-1}}{((x-p)^2+q^2)^{\gamma-1}} + \cdots + \frac{M_1 x + N_1}{(x-p)^2+q^2}\\ &\quad + \cdots + \frac{K_\delta x + L_\delta}{((x-r)^2+s^2)^\delta} + \frac{K_{\delta-1}x + L_{\delta-1}}{((x-r)^2+s^2)^\delta} + \cdots + \frac{K_1 x + L_1}{(x-r)^2+s^2}.\end{aligned}$$

The fractions which appear on the right–hand side are simple fractions and the relation is the decomposition of a proper rational fraction into a sum of simple fractions.

4.2. **Integrals of the simple fractions.** Each of the partial fractions can be integrated in terms of elementary functions:

(1) $\int \frac{A}{x-a}\, dx = A \ln|x-a| + C$,

(2) $\int \frac{A}{(x-a)^\alpha}\, dx = \frac{A}{1-\alpha} \frac{1}{(x-a)^{\alpha-1}} + C, \quad (\alpha \neq 1)$,

(3) $\int \frac{Mx+N}{(x-p)^2+q^2}\, dx = \frac{M}{2} \ln\left((x-p)^2+q^2\right) + \frac{Mp+N}{q} \arctan\left(\frac{x-p}{q}\right) + C.$

(4) $\int \frac{Mx+N}{((x-p)^2+q^2)^n}\, dx = -\frac{M}{2(n-1)((x-p)^2+q^2)^{n-1}} + (Mp+N) \int \frac{1}{((x-p)^2+q^2)^n} dx.$

Example 4.3. Find $\displaystyle\int \frac{1}{x^2-5x+6}\, dx.$

Solution: Notice that $x^2 - 5x + 6 = (x-3)(x-2)$. Then

$$\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} = \frac{A(x-2)+B(x-3)}{(x-3)(x-2)}.$$

Setting $x = 2$ we get $1 = -B$, whence $B = -1$ and setting $x = 3$ we get $A = 1$. Hence

$$\int \frac{1}{x^2-5x+6}\, dx = \ln|x-3| - \ln|x-2| + C.$$

Example 4.4. Find $\displaystyle\int \frac{1}{x^3-8x^2}\,dx$.

SOLUTION: Notice that $x^3-8x^2 = x^2(x-8)$ and that

$$\frac{1}{x^2(x-8)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-8} = \frac{Ax(x-8)+B(x-8)+Cx^2}{x^2(x-8)}$$

if and only if

$$1 = Ax(x-8)+B(x-8)+Cx^2$$

Solving, we get $A = -1/64$, $B = -1/8$ and $C = 1/64$. Hence

$$\int \frac{1}{x^3-8x^2}\,dx = \frac{-1}{64}\int \frac{dx}{x} - \frac{-1}{8}\int \frac{dx}{x^2} + \frac{1}{64}\int \frac{dx}{x-8} = \frac{x^{-2}}{128} + \frac{x^{-3}}{24} + \frac{\ln|x-8|}{64} + C.$$

5. Integral of trigonometric functions

Integrals of the kind

$$\int R(\sin x, \cos x)dx,$$

where R is a rational function, can be reduced to the integral of a rational function of the class studied above. To this end, the universal change of variable

$$t = \tan\frac{x}{2}, \quad -\pi < x < \pi$$

is used. It can be shown that

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}.$$

Usually, this change of variable leads to cumbersome calculations There are cases where the simplification does not require the change above.

- When R is odd in $\sin x$, $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, use

  $$t = \cos x, \quad dt = -\sin x dx.$$

- When R is odd in $\cos x$, $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, use

  $$t = \sin x, \quad dt = \cos x dx.$$

- When R is even in both variables, $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, use

  $$t = \tan x.$$

  in this case

  $$\sin x = \frac{t}{\sqrt{1+t^2}}, \quad \sin x = \frac{1}{\sqrt{1+t^2}}, \quad dx = \frac{dt}{1+t^2}.$$

- Integrals of the kind $\int \sin ax \cos bx dx$, $\int \sin ax \sin bx dx$ and $\int \cos ax \cos bx dx$, where $a, b \in \mathbb{R}$, reduce to immediate integrals with the trigonometric formulae

$$2 \sin x \cos y = \sin(x - y) + \sin(x + y),$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y),$$

$$2 \cos x \cos y = \cos(x - y) + \cos(x + y).$$