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I.4 QUADRATIC FORMS

1. Definition and matrix associated to a quadratic form

Definition 1.1. A polynomial of n variables $p(x_1, \dots, x_n)$ is homogeneous of degree m if and only if

$$p(\lambda x_1, \dots, \lambda x_n) = \lambda^m p(x_1, \dots, x_n),$$

for all $\lambda \in \mathbb{R}$.

Definition 1.2. A quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ is an homogeneous polynomial of degree 2, that is

$$Q(x_1, \dots, x_n) = \sum_{i,j=1}^{n} b_{ij} x_i x_j,$$

where $b_{ij} \in \mathbb{R}$, for all $i, j \in \{1, \dots, n\}$. Note that, defining $a_{ij} = \frac{b_{ij}+b_{ji}}{2}$,

$$Q(x_1, \dots, x_n) = \sum_{i=1} a_{ij} x_i^2 + 2 \sum_{\substack{i,j=1 \\ j>i}} a_{ij} x_i x_j.$$

Definition 1.3. The matrix associated to the quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ is

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix}.$$

Note that

$$Q(x_1, \dots, x_n) = (x_1, \dots, x_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and that the matrix A is symmetric.

Proposition 1.4. *Given a quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$, there is a unique symmetric matrix such that*

$$Q(\overline{x}) = \overline{x}^t A \overline{x},$$

for all $\overline{x} \in \mathbb{R}^n$. Reciprocally, given a symmetric matrix $A \in \mathcal{M}_{n\times n}$, there is a unique quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ such that

$$Q(\overline{x}) = \overline{x}^t A \overline{x},$$

for all $\overline{x} \in \mathbb{R}^n$.

1.1. **Properties.** Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form, let $\overline{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

(1) $Q(\overline{0}) = 0$.

(2) $Q(\lambda \overline{x}) = \lambda^2 Q(\overline{x})$.

(3) $Q(-\overline{x}) = Q(\overline{x})$.

2. Classification of quadratic forms

Definition 2.1. Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form.

(1) Q is positive definite if and only if
$$Q(\overline{x}) > 0,$$
for all $\overline{x} \neq \overline{0}$.

(2) Q negative definite if and only if
$$Q(\overline{x}) < 0,$$
for all $\overline{x} \neq \overline{0}$.

(3) Q is positive semidefinite if and only if
$$Q(\overline{x}) \geq 0,$$
for all $\overline{x} \in \mathbb{R}^n$ and there is $\overline{y} \in \mathbb{R}^n$ non-null such that
$$Q(\overline{y}) = 0.$$

(4) Q is negative semidefinite if and only if
$$Q(\overline{x}) \leq 0,$$
for all $\overline{x} \in \mathbb{R}^n$ and there is $\overline{z} \in \mathbb{R}^n$ non-null such that
$$Q(\overline{z}) = 0.$$

(5) Q is indefinite if and only if there are $\overline{y}, \overline{z} \in \mathbb{R}^n$, such that
$$Q(\overline{y}) > 0, \quad Q(\overline{z}) < 0.$$

Note that any non null quadratic form is of one and only one of the classes above. The null quadratic form is a special case.

Theorem 2.2. *For every quadratic form $Q : \mathbb{R}^m \to \mathbb{R}$, there is a change of variable $\overline{y} = P\overline{x}$, with P orthonormal, such that*

$$Q(\overline{y}) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2.$$

Here, $\lambda_1, \dots, \lambda_n$ are eigenvalues of the matrix A associated to Q (possibly repeated), P is a matrix such that $A = P^t DP$, and D is a diagonal matrix associated to A.

It turns out that it suffices to know the sign of the eigenvalues of A for classifying Q.

Theorem 2.3 (Criterion of the proper values)**.** *Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form with associated matrix A and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A. Then*

(1) Q is D^+ if and only if $\lambda_i > 0$, for all $i = 1, \dots, n$.

(2) Q is D^- if and only if $\lambda_i < 0$, for all $i = 1, \dots, n$.

(3) Q is SD^+ if and only if $\lambda_i \geq 0$, for all $i = 1, \dots, n$, and there is $j \in \{1, \dots, n\}$ such that $\lambda_j = 0$.

(4) Q is SD^- if and only if $\lambda_i \leq 0$, for all $i = 1, \dots, n$, and there is $j \in \{1, \dots, n\}$such that $\lambda_j = 0$.

(5) Q is indefinite if and only if there are $i, j \in \{1, \dots, n\}$, such that $\lambda_i > 0$ and $\lambda_j < 0$.

2.1. **Descartes' Rule.** Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of a single variable x, such that all its roots are real. Let

- $r^+(p)$ be the number of positive roots of p, considering their multiplicity.
- $r^-(p)$ be the number of negative roots of p, considering their multiplicity.
- $r^0(p)$ be the multiplicity of the null root.
- $s(p)$ be the number of change of signs between the consecutive and non-null coefficients of p.

Then

$$r^+(p) = s(p), \quad r^-(p) = n - r^+(p) - r^0(p).$$

Remark 2.4. To classify a quadratic form Q, it suffices to know the sign of its eigenvalues, or roots of its characteristic polynomial. An important property of the characteristic polynomial associated to a symmetric matrix is that all its roots are real (not true for an arbitrary square matrix!). Thus, Descartes' Rule applies, and allows us to classify easily

Q once we have calculated the characteristic polynomial, without knowing explicitly the eigenvalues.

Example 2.5. Classify the quadratic form $Q(x, y, z) = x^2 + 2y^2 + z^2 + 2xz - 2yz$.

The matrix associated to Q is

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

and the characteristic polynomial $p_A(\lambda) = -\lambda^3 + 4\lambda^2 - 3\lambda - 1$. As p_A comes from a symmetric matrix, is has three real roots. As it is not homogeneous, $r^0(p_A) = 0$. There are two change of signs in p_A, thus $s(p_A) = 2$ and hence the polynomial has 2 positive roots and $3 - 2 = 1$ negative root. In consequence, two eigenvalues are positive, and one is negative and the quadratic from is indefinite.

Definition 2.6. Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$.

The principal minor of order r of A is the determinant of the submatrix of A formed by the first r rows and columns.

$$\Delta_r = \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{pmatrix}.$$

Theorem 2.7. *Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form and let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be the principal minors of the associated matrix A. Then*

- *Q is D^+ if and only if $\Delta_1 > 0$, $\Delta_2 > 0$,…,$\Delta_n > 0$.*
- *Q is D^- if and only if $\Delta_1 < 0$, $\Delta_2 > 0$,…,$(-1)^n\Delta_n > 0$.*
- *If $\Delta_n = \det A \neq 0$, and none of the two previous conditions are fulfilled, then Q is indefinite.*
- *If $\Delta_n = \det A = 0$ and $\operatorname{rank} A = p$, then it is possible to exchange columns (and the same rows at once), such that the principal minor of order p of the new matrix is different from zero. Let Δ'_1,…,Δ'_p be the principal minors of the new matrix. Then*

– *Q is SD^+ if and only if $\Delta'_1 > 0$, $\Delta'_2 > 0, \ldots, \Delta'_p > 0$.*

– *Q is SD^- if and only if $\Delta'_1 < 0$, $\Delta'_2 > 0, \ldots, (-1)^p \Delta'_p > 0$.*

– *Q is indefinite if none of the two previous conditions hold true.*

Remark 2.8. The exchange of columns and the corresponding rows in the matrix A is equivalent to exchange the role of the corresponding variables.

Example 2.9. Classify the quadratic form $Q(x, y, z) = x^2 + 2y^2 + z^2 + 2xz - 2yz$.

This quadratic form has been classified above as Indefinite with Descartes' Rule. Of course, we get the same result if we use the alternative principal minors method. The matrix associated to Q is

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

and $\Delta_1 = 1 > 0$, $\Delta_2 = 2 > 0$, $\Delta_3 = |A| = -1 < 0$.

3. Restricted quadratic forms

Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form and let $V \subseteq \mathbb{R}^n$ be given by

$$V = \{\overline{x} \in \mathbb{R}^n \,:\, B\overline{x} = \overline{0}\},$$

where $B \in \mathcal{M}_{m \times n}$, and $\operatorname{rank} B = m$. Let $Q\big|_V : V \to \mathbb{R}$ the restriction of Q to V.

Definition 3.1. We say that Q restricted to V is D^+, D^-, SD^+, SD^- or indefinite, if and only if $Q\big|_V$ is D^+, D^-, SD^+, SD^- or indefinite, respectively.

Remark 3.2. (1) If Q is D^+, then $Q\big|_V$ is D^+.

(2) If Q is D^-, then $Q\big|_V$ is D^-.

(3) If Q is SD^+, then $Q\big|_V$ may be SD^+ or D^+.

(4) If Q is SD^-, then $Q\big|_V$ may be SD^- or D^-.

(5) If Q is indefinite, then $Q\big|_V$ may be of any kind.

3.1. **Classification criterion.** Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form and let $V \subseteq \mathbb{R}^n$ be given by

$$V = \{\overline{x} \in \mathbb{R}^n \,:\, B\overline{x} = \overline{0}\},$$

where $B \in \mathcal{M}_{m\times n}$, and $\operatorname{rank} B = m < n$.

(1) By substitution. Solve $B\overline{x} = \overline{0}$ to obtain m variables that depend on $n-m$ parameters. By substituting into the expression of $Q(\overline{x})$, the restricted quadratic form $Q\big|_V$ is obtained, which is an unrestricted quadratic form of $n-m$ variables.

(2) By principal minors of the bordered matrix. Let the bordered matrix

$$A^* = \begin{pmatrix} O & B \\ B^t & A \end{pmatrix}.$$

(a) If the last $n-m$ principal minors of A^* have the same sign as $(-1)^m$, then $Q\big|_V$ is D^+.

(b) If the last $n-m$ principal minors of A^* alternate sign, starting with the sign of $(-1)^{m+1}$, then $Q\big|_V$ is D^-.

Example 3.3. Classify the quadratic form $Q(x, y, z) = x^2 + 2y^2 + z^2 + 2xz - 2yz$ restricted to the subspace V defined by the equation $4x - 2y - z = 0$ (the equation describes a plane).

The bordered matrix is

$$A^* = \begin{pmatrix} 0 & 4 & -2 & -1 \\ 4 & 1 & 0 & 1 \\ -2 & 0 & 2 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

The subspace V is given by $m = 1$ equation and the number of variables is 3, thus we study the last $3 - 1 = 2$ principal minors of A^*.

$$\begin{vmatrix} 0 & 4 & -2 \\ 4 & 1 & 0 \\ -2 & 0 & 2 \end{vmatrix} = -36, \quad \begin{vmatrix} 0 & 4 & -2 & -1 \\ 4 & 1 & 0 & 1 \\ -2 & 0 & 2 & -1 \\ -1 & 1 & -1 & 1 \end{vmatrix} = -54,$$

both have the sign of $(-1)^m = (-1)^1 = -1$, thus the quadratic form restricted to V is D^+.

An alternative to this (computationally expensive) method is to plug $z = 4x - 2y$ into Q, so we get

$$q(x, y) = Q(x, y, 4x - 2y) = x^2 + 2y^2 + (4x - 2y)^2 + 2(4x - 2y)(x - y) = 25x^2 + 10y^2 - 28xy,$$

with matrix $\begin{pmatrix} 25 & -14 \\ -14 & 10 \end{pmatrix}$, which is D^+.