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I.2 LINEAR SYSTEMS

1. Linear systems of equations

1.1. **Definitions.** A *system of m linear equations* with n *unknowns* is a set of equations of the form

$$\left.\begin{array}{rcl} a_{11}x_1+\cdots+a_{1n}x_n &=& b_1\\ &\vdots& \\ a_{m1}x_1+\cdots+a_{mn}x_n &=& b_m \end{array}\right\}$$

where $a_{ij}, b_i \in \mathbb{R}$ for all $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$.

- a_{ij} are the coefficients of the system.
- b_i are the independent terms of the system.
- x_j are the unknowns of the system

A vector $(s_1,\dots,s_n)$ is a solution of the system if satisfies the identities

$$\left.\begin{array}{rcl} a_{11}s_1+\cdots+a_{1n}s_n &=& b_1\\ &\vdots& \\ a_{m1}s_1+\cdots+a_{mn}s_n &=& b_m \end{array}\right\}$$

The matrix form of the linear system is

$$\begin{pmatrix} a_{11} & \dots & a_{1n}\\ \vdots & \dots & \vdots\\ a_{m1} & \dots & a_{mn}\end{pmatrix}\begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix} b_1\\ \vdots\\ b_m\end{pmatrix}, \quad \text{or } A\mathbf{x}=\mathbf{b},$$

where $A \in \mathcal{M}_{n\times n}$ is the matrix of coefficients of the system, $\mathbf{x} \in \mathcal{M}_{n\times 1}$ is the vector of unknowns of the system and $\mathbf{b} \in \mathcal{M}_{m\times 1}$ is the vector of independent terms of the system.

Linear systems are classified corresponding to their independent term as:

(1) *Homogeneous*, if the *independent term* $\mathbf{b}$ is the null vector.

(2) *Non homogeneous*, if the *independent term* $\mathbf{b}$ is not the null vector.

Linear systems are classified corresponding to their set of solutions as:

(1) *Inconsistent* or *overdetermined*, if the solution set is empty, that is, if the system does not admit solution. *Consistent*, if the system admits solution. In this case, the system may be

(a) *Determined*, if the solution is unique.

(b) *Underdetermined*, if the solution is multiple.

The augmented matrix or complete matrix of the system, denoted by $(A|\mathbf{b}) \in \mathcal{M}m \times (n+1)$, is the matrix which contains A in the first n columns and $\mathbf{b}$ in column $n+1$, that is

$$(A|\mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right).$$

1.2. **Geometric interpretation of linear systems of two variables.** For $m \times 2$ systems, we can give the following geometric interpretation. Each equation represents a straight line in the plane and the solutions of the system (if any) are the points of interception of the m lines. In the case of two lines, $m = 2$, the system is consistent when the two lines intercept at a single point. The system is inconsistent if the two lines are parallel and not coincide. Finally, the system is under-determined if the two lines coincide, that is, if the two equations are proportional.

Example 1.1. The system $\begin{cases} 2x + y = 5 \\ 4x + 2y = 7 \end{cases}$ is inconsistent, because both lines are parallel (they have the same slope -2) but are not coincident, as the equations are not proportional.

The system $\begin{cases} x + y = 5 \\ 4y = 8 \end{cases}$ is consistent with only one solution since they have different slopes -1 and 0, respectively. Actually, the only solution is $(3, 2)$.

The system $\begin{cases} x + y = 4 \\ 2x + 2y = 8 \end{cases}$ is underdetermined. Indeed, the equations represent the same line. One can describe the set of solutions as $\{(x, 4 - x) \mid x \in \mathbb{R}\}$.

1.3. **Rouché–Frobenius Theorem.**

Theorem 1.2 (Rouché–Frobenius Theorem)**.** *A system of m linear equations and n unknowns is*

(1) Consistent if and only if $\operatorname{rank} A = \operatorname{rank}(A|\mathbf{b})$*. Moreover, for a consistent system*

(a) *If* rank $A = n$, *then the system is determined.*

(b) *If* rank $A < n$, *then the system is underdetermined. In this case, the number of parameters needed to describe the solution set is* $n -$ rank A.

(2) *Inconsistent if and only if* rank $A <$ rank $(A|\mathbf{b})$.

Remark 1.3. Every homogeneous system is consistent, as it always admits the so called trivial solution

$$x_1 = 0, x_2 = 0, \ldots, x_n = 0.$$

Theorem 1.4. *An homogeneous system* $A\mathbf{x} = \mathbf{0}$ *with* m *equations and* n *unknowns*

- *Only admits the the trivial solution if* rank $A = n$.
- *Admits infinitely many solutions if* rank $A < n$.

Example 1.5. Study the linear system

$$\begin{cases} x + y + 2z = 1 \\ 2x + y + 3z = 2 \\ 3x + 2y + 5z = 3 \end{cases}.$$

Solution: Observe that $(A|\mathbf{b}) = \begin{pmatrix} 1 & 1 & 2 & | & 1 \\ 2 & 1 & 3 & | & 2 \\ 3 & 2 & 5 & | & 3 \end{pmatrix}$. We will find the rank of matrices A and $(A|\mathbf{b})$ simultaneously by means of elementary transformations.

$$\begin{pmatrix} 1 & 1 & 2 & | & 1 \\ 2 & 1 & 3 & | & 2 \\ 3 & 2 & 5 & | & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & | & 1 \\ 0 & -1 & -1 & | & 0 \\ 0 & -1 & -1 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & | & 1 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Thus, rank $A =$ rank$(A|\mathbf{b}) = 2$. According to the previous theorem, the system is consistent, and the number of parameters in the solution set is $3 -$ rank $A = 3 - 2 = 1$. We will show in the next section how to find the solution set.

1.4. **The method of Gauss to solve linear systems.** The method of Gauss to solve linear systems consists in finding the upper echelon form of the augmented matrix $(A|\mathbf{b})$. Then, the equivalent system is solved for this simpler equivalent system, from the last equation to the first one. Let us see this with an example.

Example 1.6. Find the solutions of the system

$$\begin{cases} x + y + 2z = 1 \\ 2x + y + 3z = 2 \\ 3x + 2y + 5z = 3. \end{cases}$$

Solution: We already know that the system is consistent under-determined and the echelon form of the augmented matrix is (see example above)

$$\begin{pmatrix} 1 & 1 & 2 & | & 1 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Notice that the equivalent system is $\begin{cases} x + y + 2z = 1 \\ y + z = 0 \end{cases}$, which can be easily solved, starting by the last equation, $y = -z$ and substituting this into the first equation we get $x = 1 - y - 2z = 1 - (-z) - 2z = 1 - z$. Then, z plays the role of a parameter and the solution set is one–parametric, $\{(1 - z, -z, z) \mid z \in \mathbb{R}\}$, in accordance with our comments in the previous example.

1.5. **Cramer's Rule.** We study here a method based in determinants to give explicitly the solution for linear systems of order $n \times n$.

Theorem 1.7. *A linear system of order $n \times n$ is consistent with a unique solution if and only if $|A| \neq 0$.*

This is an easy consequence of the Rouché–Frobenius Theorem, since that when $|A| \neq 0$, $\operatorname{rank} A = n$. Since it is always true that $\operatorname{rank} A \leq \operatorname{rank}(A|\mathbf{b}) \leq n$, necessarily $\operatorname{rank} A = \operatorname{rank}(A|\mathbf{b}) = n$ and the system is consistent with a unique solution.

Let a consistent system

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = & b_1 \\ \vdots & \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = & b_n. \end{cases}$$

We know

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0.$$

Let $(x_1^*, x_2^*, \ldots, x_n^*)$ the unique solution of the system. Then, Cramer's rule gives the solution as

$$x_1^* = \frac{\begin{vmatrix} b_1 & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \ldots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}}, \quad x_2^* = \frac{\begin{vmatrix} a_{11} & b_1 & \ldots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \ldots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}}, \ldots, x_n^* = \frac{\begin{vmatrix} a_{11} & a_{12} & \ldots & b_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & b_n \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}}.$$

Example 1.8. Find the solutions of the system

$$\begin{cases} 2x - 3y + 4z = 3 \\ 4x + y + z = 6 \\ 2x - y + z = 2. \end{cases}$$

SOLUTION: $|A| = \begin{vmatrix} 2 & -3 & 4 \\ 4 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = -14 \neq 0$. Hence, by the above theorem, there is a unique solution. By Cramer's rule, it is $x^* = 1$, $y^* = 1$, and $z^* = 1$. We have used

$$\begin{vmatrix} 3 & -3 & 4 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = -14, \quad \begin{vmatrix} 2 & 3 & 4 \\ 4 & 6 & 1 \\ 2 & 2 & 1 \end{vmatrix} = -14, \quad \begin{vmatrix} 2 & -3 & 3 \\ 4 & 1 & 6 \\ 2 & -1 & 2 \end{vmatrix} = -14.$$

Continuing with this example, suppose that we have instead the system

$$\begin{cases} 2x - 3y + 4z - t = 3 \\ 4x + y + z + 2t = 6 \\ 2x - y + z + t = 2. \end{cases}$$

Since the system is not square, we cannot apply Carmer's rule directly. But we can rewrite the system as

$$\begin{cases} 2x - 3y + 4z = 3 + t \\ 4x + y + z = 6 - 2t \\ 2x - y + z = 2 - t. \end{cases}$$

The system becomes the same as above, but with different independent term. We can apply Cramer'r rule to get

$$x^* = \frac{\begin{vmatrix} 3+t & -3 & 4 \\ 6-2t & 1 & 1 \\ 2-t & -1 & 1 \end{vmatrix}}{-14} = 1 - \frac{11}{14}t,$$

$$y^* = \frac{\begin{vmatrix} 2 & 3+t & 4 \\ 4 & 6-2t & 1 \\ 2 & 2-t & 1 \end{vmatrix}}{-14} = 1 + \frac{2}{7}t,$$

$$z^* = \frac{\begin{vmatrix} 2 & -3 & 3+t \\ 4 & 1 & 6-2t \\ 2 & -1 & 2-t \end{vmatrix}}{-14} = 1 + \frac{6}{7}t.$$

Here, t plays the role of a parameter and there are infinitely many solutions. For instance,

$$(1,1,1,0), \quad \left(\frac{3}{14}, \frac{9}{7}, \frac{13}{7}, 1\right), \quad (-10,5,13,14), \quad \ldots$$

are solutions.

Example 1.9. A financial intermediary offers three portfolios X, Y, Z. Each portfolio is composed of the same three assets, A, B and C, but in different proportions. The table below shows the percentage of the assets in each of the portfolios.

	A	B	C
X	50	30	20
Y	30	70	0
Z	40	20	40

A potential costumer does not like the composition of each individual portfolio, so she asks to the intermediary if it would be possible to make a new portfolio, N, with the old portfolios, so that the participation of assets A, B and C in N be 40%, 30% and 30%, respectively. What will be the answer of the intermediary?

Solution: The intermediary must solve the following linear system to check it it admits feasible solutions.

$$\begin{cases} 50X + 30Y + 40Z = 40 \\ 30X + 70Y + 20Z = 30 \\ 20X + 0Y + 40Z = 30. \end{cases}$$

The system admits a unique solution with *non-negative* components. Using Cramer's rule or any other method, we find

$$X = \frac{1}{6}, \quad Y = \frac{1}{6}, \quad Z = \frac{2}{3}.$$

Hence, the solution is 16.66% of both portfolios X and Y, and 66.66% of Z.