February 2, 2022

I.1 MATRICES AND DETERMINANTS

1. General concepts

Definition 1.1. A matrix of m arrows and n columns is a rectangular array of real numbers

$$A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

It is said that A is of order $m \times n$.

- The ith arrow of A is formed by elements $a_{i1}, a_{i2}, \dots, a_{in}$.
- The jth column of A is formed by elements $a_{1j}, a_{2j}, \dots, a_{mj}$.
- The element (i, j) of A is a_{ij}. It belongs to the ith row and to the jth column.
- The main diagonal is formed by the elements $a_{11}, a_{22}, \dots, a_{pp}$, donde p is the lesser of n and m.

Notation 1.2. The set of all matrices of order $m \times n$ is denoted $\mathcal{M}_{m\times n}$.

Let $A, B \in \mathcal{M}_{m\times n}$, $A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$, $B = (b_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$. The matrices A and B are equal if and only if $a_{ij} = b_{ij}$ for all $i \in \{1, \dots, m\}$, for all $j \in \{1, \dots, n\}$.

2. The algebra of matrices

2.1. **Sum of matrices.** Let $A, B \in \mathcal{M}_{m\times n}$, $A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$, $B = (b_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$. The sum of A and B is $A + B = C \in \mathcal{M}_{m\times n}$, with $C = (c_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$, where $c_{ij} = a_{ij} + b_{ij}$ for all $i \in \{1, \dots, m\}$, for all $j \in \{1, \dots, n\}$.

2.2. **Properties of the sum of matrices.** Let $A, B, C \in \mathcal{M}_{m\times n}$

(1) $A + (B + C) = (A + B) + C$.

(2) $A + B = B + A$.

(3) There is $O = (0_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \in \mathcal{M}_{m\times n}$ (null matrix), such that $A + O = O + A = A$.

(4) $A + (-A) = (-A) + A = O$, where $-A = (-a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$.

2.3. **Product of an scalar and a matrix.** Let $A \in \mathcal{M}_{m\times n}$, $A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ and let $\lambda \in \mathbb{R}$. The scalar product of λ and A is $\lambda A = C \in \mathcal{M}_{m\times n}$, with $C = (c_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$, where $c_{ij} = \lambda a_{ij}$.

2.4. **Properties of the product of an scalar and a matrix.** Let $A, B \in \mathcal{M}_{m\times n}$, $\lambda, \mu \in \mathbb{R}$

(1) $\lambda(A+B) = \lambda A + \mu B$.

(2) $(\lambda+\mu)A = \lambda A + \mu A$.

(3) $(\lambda\mu A) = \lambda(\mu A)$.

(4) $1A = A$.

Example 2.1. Let $\lambda = 3$ and $A = \begin{pmatrix} 2 & 1 & 3 \\ 9 & 6 & 5 \end{pmatrix}$. Then $3A = \begin{pmatrix} 6 & 3 & 9 \\ 27 & 18 & 15 \end{pmatrix}$.

2.5. **Product of matrices.** Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{n\times p}$, where $A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$, $B = (b_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,p}}$. The product of A and B is defined as $AB = C \in \mathcal{M}_{m\times p}$, where $C = (c_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,p}}$ and

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \quad \text{for all } i \in \{1,\dots,m\},\ j \in \{1,\dots,p\}.$$

That is, the product AB is the matrix whose (i,j) element is the scalar product of the ith row of the first matrix and the j column of the second matrix, considered these as vectors.

2.6. **Properties of the product of matrices.** Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{n\times p}$, $C \in \mathcal{M}_{p\times q}$, $\lambda, \mu \in \mathbb{R}$.

(1) $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

(2) $(AB)C = A(BC)$.

(3) If $m = n = p = q$, then

$$A(B+C) = AB + AC$$
$$(B+C)A = BA + CA$$

(4) $AI_n = I_nA = A$, where

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

(5) $(\lambda A)(\mu B) = (\lambda\mu)(AB)$.

Example 2.2. Compute AB and BA, where $A = \begin{pmatrix} 2 & 1 & 5 \\ -3 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 6 \\ 7 & -4 \\ 8 & 0 \end{pmatrix}$.

ANSWER: $AB = \begin{pmatrix} 49 & 8 = 2\cdot 6 + 1\cdot(-4) + 5\cdot 0 \\ 13 & -18 \end{pmatrix}$. $BA = \begin{pmatrix} -16 & 1 & 17 \\ 26 & 7 & 27 \\ 16 & 8 & 40 \end{pmatrix}$.

2.7. **Matrix transposition.** Let $A \in \mathcal{M}_{m\times n}$, $A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$. The transpose matrix of A, denoted $A^t \in \mathcal{M}_{n\times m}$, is the matrix $A = (a'_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ with $a'_{ij} = a_{ji}$, that is

$$A^t = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

2.8. **Properties of matrix transposition.** Let $A \in \mathcal{M}_{m\times n}$, $\lambda \in \mathbb{R}$.

(1) $(A^t)^t = A$.

(2) $(\lambda A)^t = \lambda A^t$.

(3) $I_n^t = I_n$.

(4) If $B \in \mathcal{M}_{m\times n}$, then $(A+B)^t = A^t + B^t$.

(5) If $B \in \mathcal{M}_{n\times p}$, then $(AB)^t = B^tA^t$.

3. Type of Matrices

3.1. **Definitions.** 1. Row matrix: It has only one row

$$(a_{11}a_{12}\dots a_{1n}) \in \mathcal{M}_{1\times n}.$$

2. Column matrix: It has only one column

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \in \mathcal{M}_{m\times 1}.$$

3. A matrix is lower (upper) triangular if and only if all the elements above the diagonal are null: $i < j \Rightarrow a_{ij} = 0$ (all the elements below the diagonal are null: $i > j \Rightarrow a_{ij} = 0$.)

4. Square matrix of order n: it has the same number of rows and columns, $m = n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{m2} & \cdots & a_{nn} \end{pmatrix}.$$

- a_{ii}, $i = 1, \ldots, n$, are the diagonal elements.
- A is a diagonal matrix if and only if the non diagonal elements are null: $i \neq j \Rightarrow a_{ij} = 0$.
- A square matrix is scalar if and only if it is diagonal and all the diagonal elements are equal to each other.

5. $A \in \mathcal{M}_{n\times n}$ is idempotent if and only if $A^2 = A$.

6. $A \in \mathcal{M}_{n\times n}$ is nilpotent if and only if there is $p \in \mathbb{N}$ such that $A^p = O$.

7. $A \in \mathcal{M}_{n\times n}$ is symmetric if and only if $A^t = A$, that is, if $A = (a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$, then

$$a_{ij} = a_{ji} \quad \text{for all } i, j \in \{1, \ldots, n\}.$$

8. $A \in \mathcal{M}_{n\times n}$ is antisymmetric if and only if $A^t = -A$, that is, if $A = (a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$, then

$$a_{ij} = -a_{ji} \quad \text{for all } i, j \in \{1, \ldots, n\}.$$

4. Determinants

To a *square* matrix A we associate a real number called the *determinant*, $|A|$ or $\det(A)$, in the following way. The determinant is a mapping

$$\det : \mathcal{M}_{n\times n} \to \mathbb{R}$$
$$A \mapsto \det A$$

such that

- For a matrix of order 1, $A = (a)$, $\det(A) = a$.
- For a matrix of order 2, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.
- For a matrix of order 3

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

  This is known as the *expansion* of the determinant by the first column, but it can be done for any other row or column, giving the same result. Notice the sign $(-1)^{i+j}$ in front of the element a_{ij}.

Before continuing with the inductive definition, let us see an example.

Example 4.1. Compute the following determinant expanding by the second column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 4 & 3 & 5 \\ 3 & 1 & 3 \end{vmatrix} = (-1)^{1+2} 2 \begin{vmatrix} 4 & 5 \\ 3 & 3 \end{vmatrix} + (-1)^{2+2} 3 \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} + (-1)^{2+3} 1 \begin{vmatrix} 1 & 1 \\ 4 & 5 \end{vmatrix}$$

$$= -2 \cdot (-3) + 3 \cdot (0) - (1) \cdot 1 = 5$$

4.1. **Definitions.** 1. A minor of a matrix A is the determinant of a submatrix which are obtained from A by deleting several rows and the same number of columns.

2. Given a square matrix $A = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$, the complementary minor of element a_{ij}, denoted M_{ij}, is the determinant of order $n-1$ which results from the deletion of the row i and the column j containing that element.

3. Given a square matrix $A = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$, the adjoint of a_{ij}, denoted A_{ij}, is the complementary minor of a_{ij}, multiplied by $(-1)^{i+j}$, that is, $A_{ij} = (-1)^{i+j} M_{ij}$.

4. The adjoint matrix of the square matrix $A = (a_{ij})_{\substack{i=1,\dots,nm \\ j=1,\dots,n}}$, denoted A^*, is the matrix whose elements are the adjoints of the elements of A, that is

$$A^* = (A_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}.$$

4.2. **Expansion of determinants by the elements of a row or a column.** Let $A = (a_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,n}} \in \mathcal{M}_{n\times}$. For $n > 3$, the determinant is defined as follows.

- Expansion by the elements of the ith row:

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}.$$

- Expansion by the elements of the jth column:

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}.$$

Example 4.2. Find the value of the determinant

$$\begin{vmatrix} 1 & 2 & 0 & 3 \\ 4 & 7 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 0 & 2 & 0 & 7 \end{vmatrix}.$$

Answer: Expanding the determinant by the third column, one gets

$$\begin{vmatrix} 1 & 2 & 0 & 3 \\ 4 & 7 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 0 & 2 & 0 & 7 \end{vmatrix} = (-1)^{3+2}2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 0 & 2 & 7 \end{vmatrix} + (-1)^{3+3}3\begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 1 \\ 0 & 2 & 7 \end{vmatrix}.$$

4.3. **Properties of the determinants.** Let $A, B \in \mathcal{M}_{n\times n}$. Let $\lambda \in \mathbb{R}$.

(1) $|A| = |A^t|$.

(2) $|\lambda A| = \lambda^n |A|$.

(3) $|AB| = |A||B|$.

(4) If in a determinant two rows (or columns) are interchanged, then the value of the determinant is changed in sign.

(5) If two rows (columns) in a determinant are identical, then the value of the determinant is zero.

(6) If all the entries in a row (column) of a determinant are multiplied by a constant λ, then the value of the determinant is also multiplied by this constant.

(7) In a given determinant, a constant multiple of the elements in one row (column) may be added to the elements of another row (column) without changing the value of the determinant.

(8) If a determinant has a line (row or column) of zeros, then the determinant is null.

5. Inverse matrix

When we want to solve an equation like $5x = 10$, we simply divide by 5, $5^{-1}5x = 5^{-1}10$, and find quite trivially the solution $x = 2$. If the equation is *matricial*

$$AX = B,$$

where A and B are given matrices and X is the unknown matrix, we wonder whether there is an object like 5^{-1} in the scalar example above, so that the matrix equation can be solved. Obviously, if such a matrix B exists, it must fulfill $BA = I_n$.

Definition 5.1. A square matrix $A \in \mathcal{M}_{n\times n}$ is called regular or invertible if there exists a matrix $B \in \mathcal{M}_{n\times n}$ such that $AB = BA = I_n$. In this case, the matrix B is called the inverse of A and it is denoted A^{-1}.

5.1. **Properties.** Let $A, B \in \mathcal{M}_{n\times n}$ and let $\lambda \in R$, $\lambda \neq 0$.

(1) A is invertible if and only if $\det A \neq 0$.

(2) If A is invertible, then A^{-1} is unique, and it is given by

$$A^{-1} = \frac{1}{\det A}(A^*)^t.$$

(3) I_n is invertible, and $I_n^{-1} = I_n$.

(4) If A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(5) If A is invertible, then λA is invertible and $(\lambda A)^{-1} = \lambda^{-1}A^{-1}$.

(6) If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(7) If A is invertible, then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Definition 5.2. $A \in \mathcal{M}_{n\times n}$ is called orthogonal if and only is it is invertible and $A^{-1} = A^t$.

6. Rank and trace

Definition 6.1. The rank of matrix $A \in \mathcal{M}_{m\times n}$, denoted rank A, is the order of the largest non–null minor of A.

Example 6.2. Find the rank of $A = \begin{pmatrix} -1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 2 & -1 & -1 & 1 \end{pmatrix}$.

ANSWER: The rank is at most 3. Instead of finding the echelon form of A let us use minors. Notice that $\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} \neq 0$ hence, the rank of A is 2 at least. If we find one non–null minor of order 3, then the rank is 3. We need to check 4 minors of order 3

$$\begin{vmatrix} -1 & 2 & 1 \\ 0 & 3 & 1 \\ 2 & -1 & -1 \end{vmatrix} = 0, \quad \begin{vmatrix} -1 & 2 & 0 \\ 0 & 3 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} = 0.$$

6.1. **Properties of the rank of a matrix.**

(1) The rank of a matrix is invariant with respect to the following operations

- Exchanging two parallel lines (rows or columns).
- Deleting a line whose elements are all null.
- Deleting a line that is a linear combination of other parallel lines.
- Multiplying all the elements of a line by a number different from zero.
- Adding to a line another parallel line multiplied by a number.

(2) If $A \in \mathcal{M}_{m\times n}$, then rank $A \leq \min\{m, n\}$.

(3) If $A \in \mathcal{M}_{n\times n}$, then A is invertible if and only if rank $A = n$.

(4) rank $I_n = n$ and rank $O = 0$.

(5) If $A \in \mathcal{M}_{m\times n}$, then rank $A =$ rank A^t.

(6) If $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{n\times p}$, then

$$\text{rank } AB \leq \min\{\text{rank } A, \text{rank } B\}.$$

Definition 6.3. Let $A \in \mathcal{M}_{n\times n}$, where $(a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$. The trace of A, denoted by trace A, is the sum of the diagonal elements of A

$$\text{trace } A = a_{11} + \cdots + a_{nn}.$$

6.2. **Properties.** Let $A, B \in \mathcal{M}_{n\times n}$, let $\lambda \in \mathbb{R}$.

(1) trace $A^t =$ trace A.

(2) trace $\lambda A = \lambda$ trace A.

(3) trace $A + B =$ trace $A +$ trace B.

(4) trace $AB =$ trace BA.

7. Elementary operations with matrices

Definition 7.1. The following operations with rows and columns of a matrix $A \in \mathcal{M}_{m\times n}$ are called elementary operations.

- Interchange of parallel lines of A (rows or columns).
- Multiplication of a line of A (row or column) by a constant different from zero.
- Addition to a line of A (row or column) a constant multiple of another parallel line.

Definition 7.2. Two matrices $A, B \in \mathcal{M}_{m\times n}$ are called equivalent if and only if one of them can be obtained from the other by means of finitely many elementary operations.

We are interested in computing the inverse of a regular matrix by means of elementary operations.

Theorem 7.3. *If $A \in \mathcal{M}_{m\times n}$ is invertible, then A is equivalent to the identity matrix I_n.*

This says that one can find the inverse of a regular matrix by means of elementary operations on the identity matrix I_n. From a practical point of view, the *Gauss method* considers the matrix

$$(A|I_n) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & | & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & | & 0 & 0 & \dots & 1 \end{pmatrix}$$

and performs elementary operations on rows until A is transformed into the identity matrix I_n, so that I_n becomes A^{-1}.

Example 7.4. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

Answer: Consider the matrix $(A|I_3) = \begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix}$. In the following operations, r_i denotes the ith row vector, and $r_i - \lambda r_j$ means that we subtracts λ times the row

r_j to the row r_i.

$$(r_3 - r_1) \sim \begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & -1 & 1 & | & -1 & 0 & 1 \end{pmatrix}; \quad (r_3 + r_2) \sim \begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 2 & | & -1 & 1 & 1 \end{pmatrix}$$

$$(2r_2 - r_3) \sim \begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 2 & 0 & | & 1 & 1 & -1 \\ 0 & 0 & 2 & | & -1 & 1 & 1 \end{pmatrix}; \quad (2r_1 - r_2) \sim \begin{pmatrix} 2 & 0 & 0 & | & 1 & -1 & 1 \\ 0 & 2 & 0 & | & 1 & 1 & -1 \\ 0 & 0 & 2 & | & -1 & 1 & 1 \end{pmatrix}$$

$$\left(\times \frac{1}{2}\right) \sim \begin{pmatrix} 1 & 0 & 0 & | & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & | & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & | & -1/2 & 1/2 & 1/2 \end{pmatrix}$$

Hence, the inverse matrix is

$$A^{-1} = \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix}.$$

Definition 7.5. The upper echelon form of a matrix $A \in \mathcal{M}_{m\times n}$ is any of the upper triangular matrices $B = (b_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \in \mathcal{M}_{m\times n}$ which are equivalent to A and such if $b_{ii} = 0$ for some $i \in \{1, \dots, m-1\}$, then $b_{i+1,i+1} = 0$.

Theorem 7.6. *The rank of a matrix $A \in \mathcal{M}_{m\times n}$ is the number of non–null rows in any of the echelon forms A.*

Example 7.7. Find the rank of the matrix $A = \begin{pmatrix} -2 & -1 & 1 & 2 \\ 0 & 2 & 2 & -3 \\ 4 & 1 & -1 & 0 \end{pmatrix}$.

ANSWER: The rank is at most 3. Let us find the echelon form of A.

$$\begin{pmatrix} -2 & -1 & 1 & 2 \\ 0 & 2 & 2 & -3 \\ 4 & 1 & -1 & 0 \end{pmatrix} \overset{2r_1+r_3}{\sim} \begin{pmatrix} -2 & -1 & 1 & 2 \\ 0 & 2 & 2 & -3 \\ 0 & -1 & 1 & 4 \end{pmatrix} \overset{(1/2)r_2+r_3}{\sim} \begin{pmatrix} -2 & -1 & 1 & 2 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & 2 & 5/2 \end{pmatrix}.$$

Hence, the rank of A is 3 (three non–null row vectors in the echelon form of A).