Mathematics for Economics II
Question Bank 2019-2020 ordinary call

1. Block I. Algebra

1. Consider the following linear systems with two parameters $a,\ b$.

$$\begin{cases} x & +y & +az & = & -b \\ x & +ay & +z & = & 0 \\ ax & +y & +z & = & b \end{cases}$$

(a) (7 points) Classify the system.

(b) (3 points) Solve the system for $a = 2$ and $b = 3$.

Solution

By means of the operations: row2'=row2-row1, row3'=row3-a·row1 and then row3$''$=row3'+row2', we obtain the echelon system

$$\left(\begin{array}{ccc|c} 1 & 1 & a & -b \\ 0 & a-1 & 1-a & b \\ 0 & 0 & 2-a-a^2 & b(a+2) \end{array}\right).$$

Note that $2 - a - a^2 = 0$ if and only if $a = 1$ or $a = -2$.

(a)

If $a \neq 1$ and $a \neq -2$, then the system is determined (a unique solution).

If $a = 1$ and $b = 0$, then it is underdetermined (infinitely many solutions) and if $b \neq 0$, it is overdetermined (no solutions).

If $a = -2$, it is underdetermined for all b.

(b) $(3, 0, -3)$

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2. Consider the following linear systems of the two parameters $a,\ b$.

$$\begin{cases} x & +y & +az & = & -b \\ x & +ay & +z & = & -3b \\ ax & +y & +z & = & b \end{cases}$$

(a) (7 points) Classify the system.

(b) (3 points) Solve the system for $a = 2$ and $b = -4$.

Solution

By means of the operations: row2'=row2-row1, row3'=row3-a·row1 and then row3$''$=row3'+row2', we obtain the echelon system

$$\left(\begin{array}{ccc|c} 1 & 1 & a & -b \\ 0 & a-1 & 1-a & -2b \\ 0 & 0 & 2-a-a^2 & b(a-1) \end{array}\right).$$

Note that $2-a-a^2=0$ if and only if $a=1$ or $a=-2$.

(a)

If $a \neq 1$ and $a \neq -2$, then the system is determined.

If $a=1$ and $b=0$, it is underdetermined and if $a=1$ and $b \neq 0$, overdetermined.

If $a=-2$, then it underdetermined for $b=0$ and overdetermined if $b \neq 0$.

(b) $(-7, 9, 1)$

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3. Consider the matrix

$A = \begin{pmatrix} -1 & 0 & 0 \\ a & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$, where a is a parameter.

(a) (5 points) Determine the values of a for which the matrix is diagonalizable.

(b) (5 points) For the value(s) of a found in the item above, calculate the eigenvalues and eigenvectors of the matrix A and write the diagonal form of A.

Solution

(a) The characteristic polynomial is $(\lambda+1)((1-\lambda)^2-4)=0$, which yields the three eigenvalues: -1 (double) and 3 (single)

For $\lambda_1=-1$, the matrix $A-(-1)I$ becomes

$$\begin{pmatrix} 0 & 0 & 0 \\ a & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix},$$

which has rank=2 if $a \neq 0$ and rank=1 if $a=0$. Thus, A can be diagonalized iff $a=0$.

(b) For $a=0$, the eigenvectors (x,y,z) associated to $\lambda_1=-1$ verify

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

whose solutions are $\{(x,y,-y),\ x,y \in \mathbb{R}$, leading to eigenvectors $(0,1,-1)$ and $(1,0,0)$.

whereas the eigenvalues (x, y, z) associated to $\lambda_2 = 3$ satisfy the system

$$\begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

whose solutions are $\{(0, y, y),\ y \in \mathbb{R}\}$, leading to the eigenvector $(0, 1, 1)$.

A diagonal form associated to A is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

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4. Consider the matrix

$A = \begin{pmatrix} 5 & a & -2 \\ 0 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix}$, where a is a parameter.

(a) (5 points) Determine the values of a for which the matrix is diagonalizable.

(b) (5 points) For the value(s) of a found in the item above, calculate the eigenvalues and eigenvectors of the matrix A and write the diagonal form of A.

Solution

(a) The characteristic polynomial is $(5 - \lambda)((1 - \lambda)^2 - 16)$, which yields the eigenvalues 5 (double) and -3 (single).

For $\lambda = 5$, the matrix $A - \lambda I = A - 5I$ becomes

$$\begin{pmatrix} 0 & a & -2 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{pmatrix},$$

which has rank 2 if $a \neq 2$ and rank 1 if $a = 2$. Hence A is diagonalizable iff $a = 2$.

(b) Let $a = 2$. The eigenvectors (x, y, z) associated to $\lambda = 5$ satisfy the system

$$\begin{pmatrix} 0 & 2 & -2 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whose solutions are (x, y, y), $x, y \in \mathbb{R}$, leading to eigenvectors $(0, 1, 1)$ and $(1, 0, 0)$.

The eigenvectors (x, y, z) associated to $\lambda = -3$ satisfy the system

$$\begin{pmatrix} 8 & 2 & -2 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whose solutions are $(z/2, -z, z)$, $z \in \mathbb{R}$, leading to the eigenvector $(1, -2, 2)$.

A diagonal form of A is $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

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5. Consider the matrix

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & 0 & a \end{pmatrix},$$

where a a is a parameter.

(a) (5 points) Determine the values of a for which the matrix is diagonalizable.

(b) (5 points) For the value(s) of a found in the item above, calculate the eigenvalues and eigenvectors of the matrix A and write the diagonal form of A.

Solution

(a) The characteristic polynomial is $(5-\lambda)(1+\lambda)(a-\lambda)$, which leads to the eigenvalues, 5 -1 and a.

If $a \neq 5$ and $a \neq -1$, then the eigenvalues are different, thus A is diagonalizable.

When $a = 5$, 5 is double. The matrix $A - \lambda I = A - 5I$ becomes

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 3 & 0 & 0 \end{pmatrix},$$

which has rank 2, hence A is not diagonalizable.

When $a = -1$, the matrix $A - (-1)I = A + I$ becomes

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix},$$

which has rank 1, hence A is diagonalizable.

(b)

Let $a = 2$. The eigenvalues are 5, -1 and 2 and A is diagonalizable.

The eigenvectors (x, y, z) associated to 5 satisfy the system

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whose solutions are $(x, 0, x)$, $x \in \mathbb{R}$, leading to the eigenvector $(1, 0, 1)$.

The eigenvectors (x, y, z) associated to -1 satisfy

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whose solutions are $(0, y, 0)$, $y \in \mathbb{R}$, leading to the eigenvector $(0, 1, 0)$.

The eigenvectors (x, y, z) associated to 2 satisfy

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whose solutions are $(0, 0, z)$, $z \in \mathbb{R}$, leading to the eigenvector $(0, 0, 1)$.

A diagonal form of A is $\begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

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6. Consider the matrix

$$A = \begin{pmatrix} 5 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & a \end{pmatrix},$$

where a is a parameter.

(a) (5 points) Determine the values of a for which the matrix is diagonalizable.

(b) (5 points)vFor the value(s) of a found in the item above, calculate the eigenvalues and eigenvectors of the matrix A and write the diagonal form of A.

Solution

(a) The characteristic polynomial is $(5 - \lambda)(1 + \lambda)(a - \lambda)$, which leads to the eigenvalues 5 -1 and a.

When $a \neq 5$ and $a \neq -1$, the eigenvalues are different, thus A is diagonalizable.

When $a = 5$, the eigenvalue 5 is double and the matrix $A - \lambda I = A - 5I$ becomes

$$\begin{pmatrix} 0 & 0 & 3 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which has rank 2, thus A is not diagonalizable.

When $a = -1$, the matrix $A - \lambda I = A - (-1)I = A + I$ becomes

$$\begin{pmatrix} 6 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which has rank 1, hence A is diagonalizable.

(b) Let $a = -1$. The eigenvalues are 5 and -1 and the matrix is diagonalizable.

The eigenvectors (x, y, z) associated to 5 satisfy

$$\begin{pmatrix} 0 & 0 & 3 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whose solutions are $(x, 0, 0)$, $x \in \mathbb{R}$, leading to the eigenvector $(1, 0, 0)$.

The eigenvectors (x, y, z) associated to -1 sartisfy

$$\begin{pmatrix} 6 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whose solutions are $(x, y, -2x)$, $x, y \in \mathbb{R}$, leading to the eigenvectors $(1, 0, -2)$ and $(0, 1, 0)$.

A diagonal form of A is $\begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

2. Block II. Riemann Integral

1. Compute the following integrals.

(a) (4 points)

$$\int x^2 e^{x^3}\, dx.$$

(b) (6 points)

$$\iint_A xy\, dxdy,$$

where $A = \{(x, y) \, : \, 0 \leq y \leq x^2 \leq 4\}$.

Solution

(a) It is immediate, $\int x^2 e^{x^3}\, dx = \frac{1}{3} e^{x^3} + C$.

(b)

$$\iint_A xy\, dxdy = \left(\int_{-2}^{2} xdx\right)\left(\int_0^{x^2} ydy\right) = \int_{-2}^{2} xdx \frac{1}{2}\, y^2\Big|_0^{x^2} = \frac{1}{2}\int_{-2}^{2} x^5 dx = \frac{1}{2}\, \frac{x^6}{6}\Big|_{-2}^{2} = \frac{16}{3} - \frac{16}{3} = 0.$$

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2. Compute the following integrals.

(a) (4 points)

$$\int \frac{x+1}{x^2+2x} dx.$$

(b) (6 points)

$$\iint_A (x^2+y^2) dxdy,$$

where $A = \{(x, y) \, : \, 0 \leq x \leq y \leq 3\}$.

Solution

(a) It is immediate, $\frac{1}{2} \ln{(x^2 + 2x)} + C$.

(b)

$$\begin{aligned}\iint_A (x^2+y^2) dxdy &= \int_0^3 dy \int_0^y (x^2+y^2) dx = \int_0^3 dy\left(\frac{1}{3}x^3 + y^2 x\Big|_0^y\right) \\ &= \int_0^3 \left(\frac{1}{3}y^3 + y^3\right) dy = \frac{4}{3}\int_0^3 y^3 dy = 27.\end{aligned}$$

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3. Compute the following integrals.

(a) (4 points)

$$\int \frac{x}{x-1} dx.$$

(b) (6 points)

$$\iint_A e^{-(x+y)} dxdy,$$

where $A = \{(x, y) \,:\, x + y \leq 1, x \geq 0, y \geq 0\}$.

Solución

(a)

$$\int \frac{x}{x-1} dx = \int \left(1 + \frac{1}{x-1}\right) dx = x + \ln(x-1) + C.$$

(b)

$$\begin{aligned}
\iint_A e^{-(x+y)} dxdy &= \int_0^1 e^{-y} dy \int_0^{1-y} e^{-x} dx \\
&= \int_0^1 e^{-y} dy \left(-e^{-x}\Big|_0^{1-y}\right) \\
&= \int_0^1 e^{-y}(1 - e^{-(1-y)}) dy = \int_0^1 (e^{-y} - e^{-1}) dy \\
&= -e^{-y}\Big|_0^1 - e^{-1} y\Big|_0^1 = (1 - e^{-1}) - e^{-1} \\
&= 1 - \frac{2}{e}.
\end{aligned}$$

3. Block III. Improper and parametric integrals

1.

(a) (5 points) Calculate the area of the unbounded region of the plane

$A = \{(x, y) \,:\, 2 \le x,\ 0 \le y \le \frac{4}{x^2}\}$.

(b) (5 points) Let

$$F(x) = \int_{x^2}^{x} \frac{e^{xt}}{t}\, dt.$$

Compute explicitly $F'(x)$.

Solution

(a)

$$\text{area}(A) = \int_2^\infty \frac{4}{x^2} dx = -\frac{4}{x}\Big|_2^\infty = 2.$$

(b)

$$\begin{aligned} F'(x) &= \frac{e^{x^2}}{x} - \frac{e^{x^3}}{x^2}(2x) + \int_{x^2}^{x} e^{xt}\, dt = \frac{e^{x^2}}{x} - 2\frac{e^{x^3}}{x} + \frac{e^{xt}}{x}\Big|_{x^2}^{x} \\ &= \frac{e^{x^2}}{x} - 2\frac{e^{x^3}}{x} + \frac{e^{x^2}}{x} - \frac{e^{x^3}}{x} = 2\frac{e^{x^2}}{x} - 3\frac{e^{x^3}}{x} \end{aligned}$$

2.

(a) (5 points) Calculate the area of the unbounded region of the plane

$A = \{(x, y) \,:\, 1 \le x,\ 0 \le y \le \frac{2}{x^3}\}$.

(b) (5 points) Let

$$F(x) = \int_{x}^{x^2} \frac{\sin(xt)}{t}\, dt.$$

Compute explicitly $F'(x)$.

Solution

(a)

$$\text{area}(A) = \int_1^\infty \frac{2}{x^3} dx = -\frac{2}{2x^2}\Big|_1^\infty = 1.$$

(b)

$$F'(x) = (2x)\frac{\sin(x^3)}{x^2} - \frac{\sin(x^2)}{x} + \int_x^{x^2} \cos(xt)\, dt = \frac{2\sin(x^3)}{x} - \frac{\sin(x^2)}{x} + \frac{\sin(xt)}{x}\Big|_x^{x^2}$$

$$= \frac{2\sin(x^3)}{x} - \frac{\sin(x^2)}{x} + \frac{\sin(x^3)}{x} - \frac{\sin(x^2)}{x}$$
$$= \frac{3\sin(x^3)}{x} - \frac{2\sin(x^2)}{x}.$$

3.

(a) (5 points) Calculate the area of the unbounded region of the plane

$A = \{(x, y) : 0 \le x \le 4,\ 0 \le y \le \frac{4}{\sqrt{x}}\}$.

(b) (5 points) Let

$$F(x) = \int_x^{x^2} \frac{\cos(xt)}{t}\,dt.$$

Compute explicitly $F'(x)$.

Solution

(a)

$$\text{area}(A) = \int_{0^+}^4 \frac{4}{\sqrt{x}}dx = 8\sqrt{x}\Big|_0^4 = 16.$$

(b)

$$F'(x) = (2x)\frac{\cos(x^3)}{x^2} - \frac{\cos(x^2)}{x} + \int_x^{x^2} -\sin(xt)\,dt = \frac{2\cos(x^3)}{x} - \frac{\cos(x^2)}{x} + \frac{\cos(xt)}{x}\Big|_x^{x^2}$$
$$= \frac{2\cos(x^3)}{x} - \frac{\cos(x^2)}{x} + \frac{\cos(x^3)}{x} - \frac{\cos(x^2)}{x}$$
$$= \frac{3\cos(x^3)}{x} - \frac{2\cos(x^2)}{x}.$$

1.

(a) (5 points) Calculate the limit

$\lim_{n\to\infty} \cos\left(\pi \frac{1-\sqrt{1+n^2}}{n}\right)$.

(b) (5 points) Calculate the value of the series

$\sum_{n=6}^{\infty}(e^{-(n+1)} - e^{-n})$.

Solution

(a)

$$\frac{1-\sqrt{1+n^2}}{n} = \frac{1-\sqrt{1+n^2}}{n} \times \frac{1+\sqrt{1+n^2}}{1+\sqrt{1+n^2}} = \frac{-n^2}{n(1+\sqrt{1+n^2})} = \frac{-1}{\frac{1}{n}+\sqrt{\frac{1}{n^2}+1}}$$

converges to -1 as $n\to\infty$, thus the limit is $\cos(-\pi) = -1$.

(b) It is a telescoping series $\sum(b_{n+1}-b_n)$, where $b_n = e^{-n}$ converges to 0, thus the series is convergent and its sum is the first element of the series, $-e^{-6}$.

2.

(a))5 points) Calculate the limit

$\lim_{n\to\infty} \sin\left(\pi \frac{\sqrt{\frac{1}{n}+1}-1}{\frac{1}{n}}\right)$.

(b) (5 points) Prove that the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{3}{4^n}$$

is convergent and compute the error made by approximating the true value of the series with the 3 first terms of the sum ($n = 1, 2, 3$).

Solution

(a)

$$\frac{\sqrt{\frac{1}{n}+1}-1}{\frac{1}{n}} = \frac{\sqrt{\frac{1}{n}+1}-1}{\frac{1}{n}} \times \frac{\sqrt{\frac{1}{n}+1}+1}{\sqrt{\frac{1}{n}+1}+1} = \frac{\frac{1}{n}}{\frac{1}{n}\sqrt{\frac{1}{n}+1}+1} = \frac{1}{\sqrt{\frac{1}{n}+1}+1}$$

converges to $1/2$ as $n\to\infty$, thus the limit is $\sin\pi/2 = 1$.

(b)

The series converges since it is an alternate series, which general term is decreasing and con verges to 0 (Leibniz Theorem). The theorem also establishes

$$|S - S_3| < a_4,$$

where S is the sum of the series and S_3 is the sum of the first three terms. Hence

$$S_3 = a_1 + a_2 + a_3 = \frac{3}{4} - \frac{3}{16} + \frac{3}{64} = \frac{39}{64} \qquad a_4 = \frac{3}{256} \qquad |S - S_3| < \frac{3}{256} \approx 0.0117...$$

3.

(a) (5 points) Calculate the limit

$$\lim_{n\to\infty} \left(1 - \frac{2n}{n^2+1}\right)^{2n}.$$

(b) (5 points) Analyze the convergence of the series

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

Solution

(a)

$$\left(1 - \frac{2n}{n^2+1}\right)^{2n} = \left[\left(1 - \frac{1}{\frac{n^2+1}{2n}}\right)^{\frac{n^2+1}{2n}}\right]^{\frac{4n^2}{n^2+1}}$$

converges to e^{-4} as $n \to \infty$.

(b)

It suffices to apply the root test.

$$\sqrt[n]{a_n} = \sqrt[n]{ne^{-n^2}} = \sqrt[n]{n}\,\sqrt[n]{e^{-n^2}} = n^{1/n}e^{\frac{-n^2}{n}} = n^{1/n}e^{-n}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} n^{1/n}e^{-n} = 0 < 1.$$