University Carlos III of Madrid

**Department of Economics**
**Final Exam. Mathematics II. September 2002.**

| **Surname:** | | **Name:** |
|---|---|---|
| **DNI:** | **Major:** | **Group:** |

(1) Consider the following subspace of $\mathbb{R}^3$, **1 point**

$$S = \{(x, y, z) \in \mathbb{R}^3 : 2x + y + 3z = 0, x - y + z = 0, 4x - y + 5z = 0\}$$

(a) Find the dimension of $S$.
(b) Find a basis of $S$.

(2) Consider the system of equations, **1 point**

$$\left.\begin{array}{rcl} ax + y + z & = & 1 \\ x + ay + z & = & b \\ x + y + az & = & 1 \end{array}\right\}$$

(a) Discuss the system depending on the values of $a$ and $b$.
(b) Solve the system in case $a = -2$, $b = -2$.

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(3) Consider the following matrix, **1.5 points**

$$A = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Compute the characteristic polynomial and the eigenvalues.
(b) Show that the matrix $A$ is diagonalizable.
(c) Compute the diagonal form and the matrix change of basis .

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(4) Consider the set $A = \{(x, y) : x^2 + y^2 \leq 2, y \geq \sqrt{x}, x \geq 0\}$. **1.5 points**

(a) Draw the set $A$ and its boundary. Compute the intersection points of the curves defining $A$.

(b) Is the set $A$ convex? Justify your answer.

(c) Consider the function $f : A \to \mathbb{R}$ defined as

$$f(x, y) = \ln\left((x-2)^2 + (y-2)^2\right)$$

Discuss if $f$ reaches a maximum or a minimum on $A$. State the theorems you use.

(5) Consider the function $f(x, y) = ax^2 + by^2 + cxy + d$. **1 point**

(a) Assuming that $\nabla f(1, 0) = (2, 4)$, compute the values of $a$ y $c$.

(b) Assuming, in addition, that the directional derivative at the point $(0, 1)$ according to the vector $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is $D_u f(0, 1) = 4\sqrt{2}$, compute, if possible, the values $b$ and $d$.

(6) Let $f(x,y) = e^{a(x+y)} + e^{b(x-y)}$. **1.5 points**

(a) Find the Hessian, $\mathrm{H}\, f(0,0)$ of $f$ at the point $(0,0)$.

(b) Discuss for what values of $a$ and $b$ the Hessian matrix $\mathrm{H}\, f(0,0)$ of $f$ at the point $(0,0)$, is positive definite.

(c) Compute the second order Taylor polynomial of $f$ around the point $(0,0)$, for the values $a = b = 1$.

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(7) Consider the function $f(x,y) = x^2 + y^3 - 14x - 27y$. **1'5 points**

(a) Find the critical values of $f$ and classify them.

(b) Find the largest open set $A$ in which $f$ is strictly convex and the largest open set $B$ in which $f$ is strictly concave.

(c) Consider $f : A \to \mathbb{R}$. Does $f$ attain a global maximum or minimum on $A$?

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(8) Consider the function $f(x, y) = e^{xy}$. **1 point**

(a) Write the Lagrange equations that determine the extreme points of $f$ in the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 8\}$.

(b) Compute and classify the extreme points of $f$ in the set $A$.