UNIVERSIDAD CARLOS III DE MADRID
MATHEMATICS II
EXERCISES (SOLUTIONS)

CHAPTER 1: Matrices and linear systems

1-1. Compute the following determinants:

a) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 0 & 5 \end{vmatrix}$ b) $\begin{vmatrix} 3 & -2 & 1 \\ 3 & 1 & 5 \\ 3 & 4 & 5 \end{vmatrix}$ c) $\begin{vmatrix} 1 & 2 & 4 \\ 1 & -2 & 4 \\ 1 & 2 & -4 \end{vmatrix}$

Solution: subtract solution to a) is -15, to part b) is -36 and to part c) is 32.

1-2. Use that $\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = 25$, to compute the value of $\begin{vmatrix} 2a & 2c & 2b \\ 2u & 2w & 2v \\ 2p & 2r & 2q \end{vmatrix}$

Solution: In the determinant,

$$\begin{vmatrix} 2a & 2c & 2b \\ 2u & 2w & 2v \\ 2p & 2r & 2q \end{vmatrix}$$

we take out the common number 2 in each row,

$$2^3 \begin{vmatrix} a & c & b \\ u & w & v \\ p & r & q \end{vmatrix}$$

Now swap columns 2 and 3

$$-2^3 \begin{vmatrix} a & b & c \\ u & v & w \\ p & q & r \end{vmatrix}$$

Finally, swap rows 2 and 3,

$$2^3 \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = 2^3 \times 25 = 200$$

1-3. Verify the following identities without expanding the determinants:

a) $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$ b) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

Solution: a): In the determinant

$$\left|\begin{pmatrix} bc & a & a^2 \\ ac & b & b^2 \\ ab & c & c^2 \end{pmatrix}\right|$$

we multiply and divide by a to obtain

$$\frac{1}{a}a \begin{vmatrix} bc & a & a^2 \\ ac & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \frac{1}{a} \begin{vmatrix} abc & a^2 & a^3 \\ ac & b & b^2 \\ ab & c & c^2 \end{vmatrix}$$

Now, we do the same procedure with b, and we observe that the determinant is the same as

$$\frac{1}{ab} \begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ ab & c & c^2 \end{vmatrix}$$

likewise with c,

$$\frac{1}{abc} \begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$$

Taking out abc in the first column, the last expression equals

$$\frac{1}{abc} abc \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

b): Adding the second column to the third,

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix}$$

we obtain

$$\begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0$$

since columns 1 and 3 are the same.

1-4. Solve the following equation, using the properties of the determinants.

$$\begin{vmatrix} a & b & c \\ a & x & c \\ a & b & x \end{vmatrix} = 0$$

Solution: We must assume that $a \neq 0$. Otherwise, the determinant is 0 for every real number x. Since,

$$\begin{vmatrix} a & b & c \\ a & x & c \\ a & b & x \end{vmatrix} = a \begin{vmatrix} 1 & b & c \\ 1 & x & c \\ 1 & b & x \end{vmatrix}$$

the statement is equivalent (assuming $a \neq 0$) to

$$\begin{vmatrix} 1 & b & c \\ 1 & x & c \\ 1 & b & x \end{vmatrix} = 0$$

We subtract the first row to the second and third rows to obtain the following equation, ecuación

$$0 = \begin{vmatrix} 1 & b & c \\ 0 & x-b & 0 \\ 0 & 0 & x-c \end{vmatrix} = (x-b)(x-c)$$

Hence, the solutions are $x = b$ y $x = c$.

1-5. Simplify and compute the following expressions.

$$\text{a)} \begin{vmatrix} ab & 2b^2 & -bc \\ a^2c & 3abc & 0 \\ 2ac & 5bc & 2c^2 \end{vmatrix} \qquad \text{b)} \begin{vmatrix} x & x & x & x \\ x & a & a & a \\ x & a & b & b \\ x & a & b & c \end{vmatrix} \qquad \text{c)} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

Solution: a): In the determinant

$$\begin{vmatrix} ab & 2b^2 & -bc \\ a^2c & 3abc & 0 \\ 2ac & 5bc & 2c^2 \end{vmatrix}$$

we take out a in the first column, b in the second one and c in the third one,

$$abc \begin{vmatrix} b & 2b & -b \\ ac & 3ac & 0 \\ 2c & 5c & 2c \end{vmatrix}$$

and now we take out b in the first row, a in the second one and c in the third row,

$$a^2b^2c^2 \begin{vmatrix} 1 & 2 & -1 \\ c & 3c & 0 \\ 2 & 5 & 2 \end{vmatrix}$$

Finally, we take out c in the second row

$$a^2b^2c^3 \begin{vmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ 2 & 5 & 2 \end{vmatrix} = 3a^2b^2c^3$$

b): In the determinant

$$\left|\begin{pmatrix} x & x & x & x \\ x & a & a & a \\ x & a & b & b \\ x & a & b & c \end{pmatrix}\right|$$

we take out x in the first row

$$x\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & a & a \\ x & a & b & b \\ x & a & b & c \end{vmatrix}$$

Now we subtract the first row times x, from the other rows,

$$x\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a-x & a-x & a-x \\ 0 & a-x & b-x & b-x \\ 0 & a-x & b-x & c-x \end{vmatrix}$$

Expand now the determinant using the first column,

$$x\begin{vmatrix} a-x & a-x & a-x \\ a-x & b-x & b-x \\ a-x & b-x & c-x \end{vmatrix}$$

Subtract the first row from the other rows

$$x\left|\begin{pmatrix} a-x & a-x & a-x \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{pmatrix}\right|$$

we take out $a-x$ in the first row

$$x(a-x)\begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{vmatrix}$$

Expand now the determinant using the first column,

$$x(a-x)\begin{vmatrix} b-a & b-a \\ b-a & c-a \end{vmatrix} = x(a-x)(b-a)\begin{vmatrix} 1 & 1 \\ b-a & c-a \end{vmatrix} = x(a-x)(b-a)(c-b)$$

where we have taken out $b-a$ in the first row.

c): In the determinant

$$\left|\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}\right|$$

we subtract the last row from rows 2 and 3 and we expand it using the first column

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix}$$

we add now the first row to the second one and expand using the first column again,

$$-\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -(1+2) = -3$$

1-6. Let A be a square matrix of order $n \times n$ such that $a_{ij} = i + j$. Compute $|A|$.

Solution: We have to compute the determinant

$$\begin{vmatrix} 2 & 3 & 4 & \cdots & n+1 \\ 3 & 4 & 5 & \cdots & n+2 \\ 4 & 5 & 6 & \cdots & n+3 \\ 5 & 6 & 7 & \cdots & n+4 \\ 6 & 7 & 8 & \cdots & n+4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & n+3 & \cdots & 2n \end{vmatrix}.$$

If $n = 2$, this reduces to

$$\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1.$$

Suppose now that $n \geq 3$. We subtract the second row to the third one

$$\begin{vmatrix} 2 & 3 & 4 & \cdots & n+1 \\ 3 & 4 & 5 & \cdots & n+2 \\ 1 & 1 & 1 & \cdots & 1 \\ 5 & 6 & 7 & \cdots & n+4 \\ 6 & 7 & 8 & \cdots & n+4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & n+3 & \cdots & 2n \end{vmatrix}$$

Now subtract the first row to the second one

$$\begin{vmatrix} 2 & 3 & 4 & \cdots & n+1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 5 & 6 & 7 & \cdots & n+4 \\ 6 & 7 & 8 & \cdots & n+4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & n+3 & \cdots & 2n \end{vmatrix} = 0$$

since rows 2 and 3 are equal.

1-7. Let A be a square matrix of order $n \times n$ such that $A^t A = I$. Show that $|A| = \pm 1$.

Solution: Note that $1 = |I| = |A^t A| = |A^t||A| = |A|^2$. Hence, $|A|^2 = 1$ and the only possible values for the determinant are $|A| = 1$ o $|A| = -1$.

1-8. Find the rank of the following matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{pmatrix}$$

Solution: The rank

$$\operatorname{rank} A = \operatorname{rank} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 3 \end{pmatrix}$$

is the same as

$$\operatorname{rank} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & -3 & -1 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 2.$$

On the other hand,

$$\operatorname{rank} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -4 & -1 \\ 0 & 1 & 4 & 2 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -4 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 3$$

1-9. Study the rank of the following matrices, depending on the possible values of x.

$$A = \begin{pmatrix} x & 0 & x^2 & 1 \\ 1 & x^2 & x^3 & x \\ 0 & 0 & 1 & 0 \\ 0 & 1 & x & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \qquad C = \begin{pmatrix} x & -1 & 0 & 1 \\ 0 & x & -1 & 1 \\ 1 & 0 & -1 & 2 \end{pmatrix}$$

Solution: We compute first the rank of A.

$$\left| \begin{pmatrix} 1 & x^2 & x^3 \\ 0 & 0 & 1 \\ 0 & 1 & x \end{pmatrix} \right| = \begin{vmatrix} 0 & 1 \\ 1 & x \end{vmatrix} = -1$$

so $\operatorname{rank} A \geq 3$ for any value of x. Expanding the determinant of A using the third row

$$|A| = \begin{vmatrix} x & 0 & 1 \\ 1 & x^2 & x \\ 0 & 1 & 0 \end{vmatrix}$$

Now we expand the determinant using row 3

$$|A| = -\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} = -(x^2 - 1)$$

so the rank of A is 4 if $x^2 \neq 1$, that is if $x \neq 1$ and $x \neq -1$. To sum up,

$$\operatorname{rank} A = \begin{cases} 3, & \text{if } x = 1 \text{ o } x = -1; \\ 4, & \text{in all other cases.} \end{cases}$$

Now we compute the rank of B. Note that the minor

$$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$$

does not vanish. Hence, $\operatorname{rank} B \geq 2$. On the other hand,

$$\operatorname{rank} B = \operatorname{rank} \left(\begin{pmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \right) = \operatorname{rank} \begin{pmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 0 & 1 & 5 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & x & x^2 \\ 0 & 2-x & 4-x^2 \\ 0 & 1 & 5 \end{pmatrix}$$

and this rank is 3, unless $5(2-x) = 4 - x^2$. This happens if and only if $x^2 - 5x + 6 = 0$, that is if

$$x = \frac{5 \pm \sqrt{25 - 24}}{2} = 2, 3$$

To sum up,

$$\operatorname{rank} B = \begin{cases} 2, & \text{si } x = 2 \text{ o } x = 3; \\ 3, & \text{en los demás casos.} \end{cases}$$

Finally, we compute the rank of C.

$$\operatorname{rank} C = \operatorname{rank} \begin{pmatrix} x & -1 & 0 & 1 \\ 0 & x & -1 & 1 \\ 1 & 0 & -1 & 2 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & 0 & -1 & 2 \\ x & -1 & 0 & 1 \\ 0 & x & -1 & 1 \end{pmatrix} =$$

$$\operatorname{rank} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & x & 1-2x \\ 0 & x & -1 & 1 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & x & 1-2x \\ 0 & 0 & x^2-1 & 1+x-2x^2 \end{pmatrix}$$

ant this rank es 3 unless $x^2 - 1 = 0$ and $1 + x - 2x^2 = 0$. The solutions to $x^2 - 1 = 0$ are $x = 1$ and $x = -1$. The solutions to $2x^2 - x - 1$ are

$$x = \frac{1 \pm \sqrt{1+8}}{4} = 1, -\frac{1}{2}$$

Hence, $x = 1$ is the only solution to both equations. Therefore,

$$\operatorname{rank} C = \begin{cases} 2, & \text{if } x = 1; \\ 3, & \text{otherwise.} \end{cases}$$

1-10. Let A and B be square, invertible matrices of the same order. Solve for X in the following equations.

(a) $X^t \cdot A = B$.

(b) $(X \cdot A)^{-1} = A^{-1} \cdot B$.

Solve for X in the preceding equations, when $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Solution: a): Taking transpose in the equation $X^tA = B$, we obtain $B^t = (X^tA)^t = A^tX$. Solving for X we have that $X = (A^t)^{-1}B^t = (A^{-1})^tB^t$. When

$$A = \left(\left(\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}\right)\right) \qquad B = \left(\left(\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right)\right)$$

we see that

$$X = (A^{-1})^tB^t = \begin{pmatrix} 6 & 1 & 1 \\ -9 & -2 & -1 \\ -2 & 0 & 0 \end{pmatrix}$$

b): Taking inverse matrices in the equation $(XA)^{-1} = A^{-1}B$ we see that $XA = (A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A$. And since, A is invertible, we can solve for $X = B^{-1}AA^{-1} = B^{-1}$. When

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

we obtain

$$X = B^{-1} = \begin{pmatrix} -1/2 & 1/2 & 1/2 \\ 0 & -1 & 1 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$$

1-11. Whenever possible, compute the inverse of the following matrices.

$$A = \begin{pmatrix} 1 & 0 & x \\ -x & 1 & -\frac{x^2}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & x & 3 \\ 4 & 1 & -x \end{pmatrix}$$

Solution: We use the formula to compute the inverse of de A que

$$A^{-1} = \begin{pmatrix} 1 & 0 & -x \\ x & 1 & \frac{-x^2}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

We may also compute the inverse by performing elementary operations by rows,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & x & 1 & 0 & 0 \\ -x & 1 & -\frac{x^2}{2} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Bow we add to the second row the first one times x,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & x & 1 & 0 & 0 \\ 0 & 1 & \frac{x^2}{2} & x & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

subtract the third row times x to the first one,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -x \\ 0 & 1 & \frac{x^2}{2} & x & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Substract now the third row times $\frac{x^2}{2}$ to the second row,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -x \\ 0 & 1 & 0 & x & 1 & -\frac{x^2}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

from here we obtain the inverse matrix o A.

Now we compute the inverse matrix of

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & x & 3 \\ 4 & 1 & -x \end{pmatrix}$$

By the usual formula we see that

$$B^{-1}=\frac{1}{x^2-4x+3}\begin{pmatrix} x^2+3 & 1 & -x \\ -12 & x-4 & 3 \\ 4x & 1 & -x \end{pmatrix}$$

We also compute the inverse matrix by noticing that

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & t & 3 & 0 & 1 & 0 \\ 4 & 1 & -t & 0 & 0 & 1 \end{array}\right)$$

now we subtract from the third row the first row times 4,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & t & 3 & 0 & 1 & 0 \\ 0 & 1 & 4-t & -4 & 0 & 1 \end{array}\right)$$

exchange rows 2 and 3

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4-t & -4 & 0 & 1 \\ 0 & t & 3 & 0 & 1 & 0 \end{array}\right)$$

Add the second row times t to the third row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4-t & -4 & 0 & 1 \\ 0 & 0 & t^2-4t+3 & 4t & 1 & -t \end{array}\right)$$

From here we see that $|A|=t^2-4t+3$. The roots of this polynomial are

$$t=\frac{4\pm\sqrt{16-12}}{2}=1,3$$

so the inverse matrix exists if and only if $t\neq 1$ y $t\neq 3$. Assuming this inequality, divide by t^2-4t+3 the last row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4-t & -4 & 0 & 1 \\ 0 & 0 & 1 & \frac{4t}{t^2-4t+3} & \frac{1}{t^2-4t+3} & -\frac{t}{t^2-4t+3} \end{array}\right)$$

and add the third row to the first one and third row times $t-4$ to the second row,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{t^2+3}{t^2-4t+3} & \frac{1}{t^2-4t+3} & \frac{-t}{t^2-4t+3} \\ 0 & 1 & 0 & \frac{-12}{t^2-4t+3} & \frac{t-4}{t^2-4t+3} & \frac{3}{t^2-4t+3} \\ 0 & 0 & 1 & \frac{4t}{t^2-4t+3} & \frac{1}{t^2-4t+3} & \frac{-t}{t^2-4t+3} \end{array}\right)$$

so

$$A^{-1}=\frac{1}{t^2-4t+3}\begin{pmatrix} t^2+3 & 1 & -t \\ -12 & t-4 & 3 \\ 4t & 1 & -t \end{pmatrix}$$

1-12. Whenever possible, compute the inverse of the following matrices,

$$A=\begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}\qquad B=\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}\qquad C=\begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

Solution: We shall use Gauss' method to compute the inverse. We begin with the matrix

$$(A|\,I)=\left(\begin{array}{ccc|ccc} 4 & 6 & 0 & 1 & 0 & 0 \\ -3 & -5 & 0 & 0 & 1 & 0 \\ -3 & -6 & 1 & 0 & 0 & 1 \end{array}\right)$$

divide the first row by 4

$$\left(\begin{array}{ccc|ccc} 1 & 6/4 & 0 & 1/4 & 0 & 0 \\ -3 & -5 & 0 & 0 & 1 & 0 \\ -3 & -6 & 1 & 0 & 0 & 1 \end{array}\right)$$

now, subtract the third row to the second one

$$\left(\begin{array}{ccc|ccc} 1 & 6/4 & 0 & 1/4 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ -3 & -6 & 1 & 0 & 0 & 1 \end{array}\right)$$

add the third row to the first one times 3,

$$\left(\begin{array}{ccc|ccc} 1 & 3/2 & 0 & 1/4 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -3/2 & 1 & 3/4 & 0 & 1 \end{array}\right)$$

add the third row to the first one

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -3/2 & 1 & 3/4 & 0 & 1 \end{array}\right)$$

add the second row times 3/2 to the third row,

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1/2 & 3/4 & 3/2 & -1/2 \end{array}\right)$$

multiply the third row by -2

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -3/2 & -3 & 1 \end{array}\right)$$

add the second row to the third one and subtract them to the first one

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/2 & 3 & 0 \\ 0 & 1 & 0 & -3/2 & -2 & 0 \\ 0 & 0 & 1 & -3/2 & -3 & 1 \end{array}\right)$$

Thus, the inverse is

$$\begin{pmatrix} 5/2 & 3 & 0 \\ -3/2 & -2 & 0 \\ -3/2 & -3 & 1 \end{pmatrix}$$

Note that $|B| = 0$, so B does not have an inverse.

We compute the inverse matrix of C using Gauss' methor

$$(C|\,I) = \left(\begin{array}{ccc|ccc} 4 & 5 & -2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Multiply the third row by -1 and exchange rows 1 and 3

$$(C|\,I) = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & -1 \\ -2 & -2 & 1 & 0 & 1 & 0 \\ 4 & 5 & -2 & 1 & 0 & 0 \end{array}\right)$$

add the second row times 2 to the third row

$$(C|\,I) = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & -1 \\ -2 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \end{array}\right)$$

add the first row times 2 to the first one

$$(C|\,I) = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 & 2 & 0 \end{array}\right)$$

exchange rows 2 and 3

$$(C|\,I) = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 & -2 \end{array}\right)$$

The third row times -1,

$$(C|\,I) = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right)$$

add the row 3 to row 1,

$$(C|\,I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right)$$

subtract row 2 to row 1

$$(C|\,I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -3 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right)$$

the inverse is

$$C^{-1} = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

1-13. Given the system $\left\{\begin{array}{l} mx - y = 1 \\ x - my = 2m - 1 \end{array}\right.$ compute m so that the system,

(a) has no solution,
(b) has infinitely many solutions,
(c) has a unique solution; and
(d) has a solution with $x = 3$.

Solution: The matrix associated to the system is

$$\left(\begin{array}{cc|c} m & -1 & 1 \\ 1 & -m & 2m-1 \end{array}\right)$$

whose rank is the same as

$$\text{rank}\left(\begin{array}{cc|c} 1 & -m & 2m-1 \\ m & -1 & 1 \end{array}\right) = \text{rank}\left(\begin{array}{cc|c} 1 & -m & 2m-1 \\ 0 & m^2-1 & 1+m-2m^2 \end{array}\right)$$

Therefore, the rank of this matrix 2 is $m^2 \neq 1$. That is, if $m \neq 1$ and $m \neq -1$ then the system has a unique solution. In this case the system is equivalent to the following one

$$\begin{aligned} x - my &= 2m - 1 \\ (m^2-1)y &= 1 + m - 2m^2 \end{aligned}$$

and the solution is

$$\begin{aligned} y &= \frac{1+m-2m^2}{m^2-1} = \frac{-(m-1)(1+2m)}{(m-1)(m+1)} = \frac{-1-2m}{m+1} \\ x &= 2m - 1 + my = 2m - 1 - m\frac{1+2m}{m+1} = \frac{-1}{m+1} \end{aligned}$$

If $x = 3$ is a solution we must have that

$$3 = \frac{-1}{m+1}$$

this implies that $m = -4/3$.

We study now the case $m = 1$.

$$\text{rank}\left(\begin{array}{cc|c} 1 & -m & 2m-1 \\ 0 & m^2-1 & 1+m-2m^2 \end{array}\right) = \text{rank}\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right) = 1 = \text{rank}\, A$$

and the system has a unique solution. The original system of equations is equivalent to the following one

$$x - y = 1$$

and the set of solutions is $\{1 + y, y) : y \in R\}$. Taking $y = 2$, we obtain the solution $(3, 2)$.

For the case $m = -1$ we have that

$$\text{rank}\left(\begin{array}{cc|c} 1 & -m & 2m-1 \\ 0 & m^2-1 & 1+m-2m^2 \end{array}\right) = \text{rank}\left(\begin{array}{cc|c} 1 & 1 & -3 \\ 0 & 0 & -2 \end{array}\right) = 2 \neq \text{rank}\, A$$

and the system is inconsistent.

1-14. Given the system of linear equations $\left\{\begin{array}{l} x + ay = 1 \\ ax + z = 1 \\ ay + z = 2 \end{array}\right.$

(a) Express it in matrix form;
(b) Write the unknowns, the independent terms and the associated homogeneous system ;
(c) Discuss and solve it according to the values of a.

Solution: The augmented matrix is

$$(A|B) = \left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ a & 0 & 1 & 1 \\ 0 & a & 1 & 2 \end{array}\right)$$

First, we swap rows 2 and 3,

$$\operatorname{rank}(A|B) = \operatorname{rank}\left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & a & 1 & 2 \\ a & 0 & 1 & 1 \end{array}\right)$$

Now, we subtract row 1 times a from row 3

$$\operatorname{rank}(A|B) = \operatorname{rank}\left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & a & 1 & 2 \\ 0 & -a^2 & 1 & 1-a \end{array}\right)$$

We add row 2 times a to row 2,

$$\operatorname{rank}(A|B) = \operatorname{rank}\left(\begin{array}{ccc|c} 1 & a & 0 & 1 \\ 0 & a & 1 & 2 \\ 0 & 0 & 1+a & 1+a \end{array}\right)$$

From this, we see that if $a \neq 0$ y $a \neq -1$ then $\operatorname{rank}(A) = \operatorname{rank}(A|B) = 3 =$ the number of unknowns, so the system has a unique solution. In this case, the original system is equivalent to the following one,

$$\begin{aligned} x + ay &= 1 \\ ay + z &= 2 \\ (1+a)z &= 1+a \end{aligned}$$

and we obtain the solution $z = 1$, $y = 1/a$, $x = 0$.

If $a = -1$, then

$$\operatorname{rank}(A|B) = \operatorname{rank}\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

from this we see that $\operatorname{rank}(A) = \operatorname{rank}(A|B) = 2$ which is strictly less than the number of unknowns, so the system is undetermined. Now, the original system is equivalent to the following one,

$$\begin{aligned} x - y &= 1 \\ -y + z &= 2 \end{aligned}$$

We take z as the parameter and solve for the variables

$$y = z - 2 \qquad x = 1 + y = z - 1$$

subtract set of solutions is

$$\{(z-1, z-2, z) \in R^3 : z \in R\}$$

Finally, if $a = 0$, then

$$\operatorname{rank}(A|B) = \operatorname{rank}\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

and we see that $\operatorname{rank}(A) = 2 < \operatorname{rank}(A|B) = 3$, so the system is inconsistent.

1-15. Discuss and solve the following system $\left\{\begin{aligned} x + y + z + 2t - w &= 1 \\ -x - 2y + 2w &= -2 \\ x + 2z + 4t &= 0 \end{aligned}\right.$

Solution: The augmented matrix $A|B)$ is

$$(A|B) = \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & -1 & 1 \\ -1 & -2 & 0 & 0 & 2 & -2 \\ 1 & 0 & 2 & 4 & 0 & 0 \end{array}\right)$$

By performing elementary row operations we see that

$$\operatorname{rank}(A|B) = \operatorname{rank}\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 2 & 1 & -1 \\ 0 & -1 & 1 & 2 & 1 & -1 \end{array}\right) = \operatorname{rank}\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

so rank(A) = rank$(A|B)$ = 2 which is less than the number of unknowns. The system is consistent and underdetermined. Since there are five unknowns, we obtain $5 - 2 = 3$ parameters. We have to solve the following linear system,

$$\begin{aligned} x + y + z + 2t - w &= 1 \\ -y + z + 2t + w &= -1 \end{aligned}$$

We choose z, w and t as a parameters and solve for $y = z + 2t + w + 1$, $x = 1 - y - z - 2t - w = -2z - 4t$. The set of solutions is

$$\{(-2z - 4t, z + 2t + w + 1, z, t, w) \in \mathbb{R}^5 : z, t, w \in \mathbb{R}\}$$

1-16. Discuss and solve the following system according to the values of the parameters.

$$\begin{cases} x + y + z = 0 \\ ax + y + z = b \\ 2x + 2y + (a+1)\, z = 0 \end{cases}$$

Solution: The augmented matrix is

$$\operatorname{rank}(A|B) = \operatorname{rank}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ a & 1 & 1 & b \\ 2 & 2 & a+1 & 0 \end{array}\right)$$

Performing elementary row operations,

$$\operatorname{rank}(A|B) = \operatorname{rank}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1-a & 1-a & b \\ 0 & 0 & a-1 & 0 \end{array}\right)$$

If $a \neq 1$, then the system has a unique solution and is equivalent to the following one

$$\begin{aligned} x + y + z &= 0 \\ y + z &= \frac{b}{1-a} \\ (a-1)z &= 0 \end{aligned}$$

The solution is

$$z = 0, \quad y = \frac{b}{1-a}, \quad x = \frac{-b}{1-a}$$

Now, we study the linear system for the value $a = 1$. In this case,

$$\operatorname{rank}(A|B) = \operatorname{rank}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So, when $b \neq 0$ the system is inconsistent, since rank$(A) = 1$, rank$(A|B) = 2$.

Finally, if $a = 1$ and $b = 0$ then rank(A) = rank$(A|B) = 1$ and the system is consistent and underdetermined with $3 - 1 = 2$ parameters. The original system is equivalent to the following one

$$x + y + z = 0$$

Taking y and z as parameters, the set of solutions is

$$\{(-y - z, y, z) : y, z \in R\}$$

To sum up,

$$\begin{cases} a \neq 1, & \text{There is a unique solution: } z = 0, \quad y = \frac{b}{1-a}, \quad x = \frac{-b}{1-a}; \\ a = 1, & \text{If } \begin{cases} b \neq 0, & \text{inconsistent;} \\ b = 0, & \text{undetermined. The set of solutions is the set } \{(-y - z, y, z) : y, z \in R\}. \end{cases} \end{cases}$$

1-17. Discuss and solve the following system according to the values of the parameters.

$$\left\{\begin{array}{r} x-2y+bz=3 \\ 5x+2y=1 \\ ax+z=2 \end{array}\right.$$

Solution: The augmented matrix is

$$(A|B)=\left(\begin{array}{ccc|c} 1 & -2 & b & 3 \\ 5 & 2 & 0 & 1 \\ a & 0 & 1 & 2 \end{array}\right)$$

Performing elementary row operations we see that

$$\operatorname{rank}(A|B)=\operatorname{rank}\left(\begin{array}{ccc|c} 1 & -2 & b & 3 \\ 0 & 12 & -5b & -14 \\ 0 & 2a & 1-ab & 2-3a \end{array}\right)$$

We subtract row 2 times $a/6$ from row 3,

$$\operatorname{rank}(A|B)=\operatorname{rank}\left(\begin{array}{ccc|c} 1 & -2 & b & 3 \\ 0 & 12 & -5b & -14 \\ 0 & 0 & 1-\frac{ab}{6} & 2-\frac{2a}{3} \end{array}\right)$$

And we see that if $ab\neq 6$ then the unique solution is

$$z=\frac{12-4a}{6-ab},\quad y=\frac{-7}{6}+\frac{5}{12}\frac{12-4a}{6-ab}b,\quad x=3+\frac{12-4a}{6-ab}b-\frac{7}{3}+\frac{5}{6}\frac{12-4a}{6-ab}b$$

If $ab=6$ we solve for $a=6/b$ (since $b\neq 0$) to obtain

$$\operatorname{rank}(A|B)=\operatorname{rank}\left(\begin{array}{ccc|c} 1 & -2 & b & 3 \\ 0 & 12 & -5b & -14 \\ 0 & 0 & 0 & \frac{2b-4}{b} \end{array}\right)$$

So, if $b=2$ (and hence $a=3$) then $\operatorname{rank}(A|B)=\operatorname{rank}(A)=2$ and the system is consistent and underdetermined. It is equivalent to the following system

$$\begin{array}{rcl} x-2y+2z & = & 3 \\ 12y-10z & = & -14 \end{array}$$

Taking z as the parameter, the set of solutions is

$$\{(\frac{2-z}{3},\frac{5z-7}{6},z):z\in R\}$$

Finally, if $b\neq 2$ then the system is inconsistent.

1-18. Using Cramer's method, solve the following system.

$$\left\{\begin{array}{r} -x+y+z=3 \\ x-y+z=7 \\ x+y-z=1 \end{array}\right.$$

Solution: The determinant of the associated matrix is

$$\left|\begin{array}{rrr} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right|=4$$

And the solutions are

$$x=\frac{1}{4}\left|\begin{array}{rrr} 3 & 1 & 1 \\ 7 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right|=\frac{16}{4}=4\quad y=\frac{1}{4}\left|\begin{array}{rrr} -1 & 3 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & -1 \end{array}\right|=\frac{8}{4}=2\quad z=\frac{1}{4}\left|\begin{array}{rrr} -1 & 1 & 3 \\ 1 & -1 & 7 \\ 1 & 1 & 1 \end{array}\right|=\frac{20}{4}=5$$

1-19. Using Cramer's method, solve the following system.

$$\begin{cases} x + y = 12 \\ y + z = 8 \\ x + z = 6 \end{cases}$$

Solution: The determinant of the associated matrix is

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$$

And the solutions are

$$x = \frac{1}{2}\begin{vmatrix} 12 & 1 & 0 \\ 8 & 1 & 1 \\ 6 & 0 & 1 \end{vmatrix} = \frac{10}{2} = 5 \quad y = \frac{1}{2}\begin{vmatrix} 1 & 12 & 0 \\ 0 & 8 & 1 \\ 1 & 6 & 1 \end{vmatrix} = \frac{14}{2} = 7 \quad z = \frac{1}{2}\begin{vmatrix} 1 & 1 & 12 \\ 0 & 1 & 8 \\ 1 & 0 & 6 \end{vmatrix} = \frac{2}{2} = 1$$

1-20. Using Cramer's method, solve the following system.

$$\begin{cases} x + y - 2z = 9 \\ 2x - y + 4z = 4 \\ 2x - y + 6z = -1 \end{cases}$$

Solution: The determinant of the associated matrix is

$$\begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & 4 \\ 2 & -1 & 6 \end{vmatrix} = -6$$

And the solutions are

$$x = \frac{1}{-6}\begin{vmatrix} 9 & 1 & -2 \\ 4 & -1 & 4 \\ 1 & -1 & 6 \end{vmatrix} = \frac{-36}{-6} = 6 \quad y = \frac{1}{-6}\begin{vmatrix} 1 & 9 & -2 \\ 2 & 4 & 4 \\ 2 & 1 & 6 \end{vmatrix} = \frac{12}{-6} = -2 \quad z = \frac{1}{-6}\begin{vmatrix} 1 & 1 & 9 \\ 2 & -1 & 4 \\ 2 & -1 & 1 \end{vmatrix} = \frac{15}{-6} = \frac{-5}{2}$$

1-21. Given the following system of two equations with three unknowns

$$\begin{cases} x + 2y + z = 3 \\ ax + (a+3)y + 3z = 1 \end{cases}$$

(a) Study for what values is of a the system is not compatible.
(b) For each value of the parameter a, for which the system is compatible, write the general solution.

Solution: The rank of the associated matrix is

$$\operatorname{rank}(A|B) = \operatorname{rank}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ a & a+3 & 3 & 1 \end{array}\right) = \operatorname{rank}\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3-a & 3-a & 1-3a \end{array}\right)$$

We see that if $a = 3$ then the system has no solutions because $\operatorname{rank}(A|B) = 2$, $\operatorname{rank}(A) = 1$. If $a \neq 3$ the system is consistent and underdetermined. It is equivalent to the following one

$$\begin{aligned} x + 2y + z &= 3 \\ (3-a)y + (3-a)z &= 1 - 3a \end{aligned}$$

Taking z as the parameter, the set of solutions is

$$\{(\frac{7+3a}{3-a} + z, \frac{1-3a}{3-a} - z, z) : z \in R\}$$

1-22. Given the homogeneous system

$$\begin{cases} 3x + 3y - z = 0 \\ -4x - 2y + mz = 0 \\ 3x + 4y + 6z = 0 \end{cases}$$

(a) Compute m so that it has no trivial solutions and
(b) solve it for that value.

Solution: The rank of the associated matrix is

$$\operatorname{rank}\begin{pmatrix} 3 & 3 & -1 \\ -4 & -2 & m \\ 3 & 4 & 6 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 3 & 3 & -1 \\ -1 & 1 & m-1 \\ 0 & 1 & 7 \end{pmatrix}$$

we swap the first two rows

$$= \operatorname{rank}\begin{pmatrix} -1 & 1 & m-1 \\ 3 & 3 & -1 \\ 0 & 1 & 7 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -1 & 1 & m-1 \\ 0 & 6 & 3m-4 \\ 0 & 1 & 7 \end{pmatrix}$$

and now swap the last two rows

$$= \operatorname{rank}\begin{pmatrix} -1 & 1 & m-1 \\ 0 & 1 & 7 \\ 0 & 6 & 3m-4 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -1 & 1 & m-1 \\ 0 & 1 & 7 \\ 0 & 0 & 3m-46 \end{pmatrix}$$

The system has non-trivial solution if and only if the determinant vanishes. This happens for the value $m = 46/3$. In this case, the original system is equivalent to the following one

$$\begin{aligned} -x + y + \frac{43}{3}z &= 0 \\ y + 7z &= 0 \end{aligned}$$

and the set of solutions is

$$\{(22z/3, -7z, z) : z \in \mathbb{R}\}$$