**Department of Economics**
**Mathematics II. Final Exam. September 2007.**

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| **Last Name:** | | **Name:** |
| **DNI:** | **Degree:** | **Group:** |

**IMPORTANT**

- **DURATION OF THE EXAM: 2h. 30min.**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page. **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.
- Read the exam carefully. Each part of the exam counts $0'5$ points. Please, check that there are 14 pages in this booklet

| Problem | Points |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| Total | |

(1) Consider the following system of linear equations,

$$\begin{aligned} x - y - bz &= 1 \\ x + (a-1)y &= b+1 \\ x + (a-1)y + (a-b)z &= 2b+1 \end{aligned}$$

where $a, b \in \mathbb{R}$ are parameters.

(a) Classify the system according to the values of the parameters $a, b$.

(b) Solve the above system for the values of $a$ and $b$ for which the system is underdetermined. How many parameters are needed to describe the solution?

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(2) The matrix

$$A = \begin{pmatrix} 7 & 1 & -1 \\ 2 & 8 & -2 \\ 1 & 1 & 5 \end{pmatrix}$$

has $\lambda_1 = 6$ and $\lambda_2 = 8$ as eigenvalues. (You do not need to prove this). Solve the following.

(a) Find the eigenvectors of the matrix $A$.

(b) Justify whether the matrix $A$ is diagonalizable. And, if so, find two matrices $D$ and $P$ such that $A = PDP^{-1}$.

(c) Find a matrix $B$ such that $B^2 = A$. It is enough to write $B$ as the product of three matrices.

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(3) Given the linear map $f : \mathbb{R}^4 \to \mathbb{R}^2$,

$$f(x, y, z) = (x - y + z - t, -x + y - z + t)$$

(a) Compute the dimensions of the kernel and the image and find some equations defining these subspaces.

(b) Find a basis of the image of $f$ and a basis of the kernel of $f$.

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(4) Given the set

$$A = \{(x, y) \in \mathbb{R}^2 : \quad |y| \leq x, \quad x \leq 1\}$$

(a) Draw the set $A$, computing its vertices. Draw its boundary and interior and discuss whether the set $A$ is open, closed, bounded, compact and/or convex. You must explain your answer.

(b) Consider the function

$$f(x, y) = \frac{1}{(2x-1)^2 + (y-1)^2}$$

¿At what point(s) does the function $f$ fail to be continuous? Determine if $f$ attains a maximum and a minimum on the set $A$. State the theorems that you are using.

(c) Draw the level curves of the function $g(x, y) = (x-2)^2 + y^2$ and use them to determine the maxima and minima of $g$ on $A$.

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(5) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = \begin{cases} \frac{xy}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Study if the function $f$ is continuous at the point $(0, 0)$. Study at which points of $\mathbb{R}^2$ the function $f$ is continuous.

(b) Compute the partial derivatives of $f$ at the point $(0, 0)$.

(c) At which points of $\mathbb{R}^2$ is the function $f$ differentiable?

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(6) Given the quadratic form

$$Q(x, y, z) = 2x^2 + y^2 + 3z^2 + 4axy + 2yz$$

(a) Determine the matrix associated to the above quadratic form.

(b) Classify the quadratic form, according to the values of the parameter $a$.

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(7) Consider the function

$$f(x, y) = x^2 - \ln(x^2) - 4\ln(y^2) + y^2$$

(a) Compute the gradient vector and the Hessian matrix of $f$ at any point $(x, y)$ in domain of the function.

(b) Determine the critical points of $f$ and classify them.

(c) Determine if the function $f$ attains any extreme points on the set

$$A = \{(x, y) \in \mathbb{R}^2 : x > 0, \quad y > 0\}$$

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(8) Consider the function

$$f(x, y, z) = x + y - y^2 - x^2 - \frac{z^2}{2}$$

and the set

$$A = \{(x, y, z) : x + y + z = 0\}$$

(a) Find the Lagrange equations that determine the extreme points of $f$ in the set $A$.

(b) Determine the points that satisfy the Lagrange equations and find the extreme points of $f$, specifying whether they correspond to a maximum or minimum.

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