Department of Economics
Mathematics II. Final Exam. September 2007

Last Name:		**Name:**
DNI:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h. 30min.**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page. **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.
- Read the exam carefully. Each part of the exam counts $0'5$ points. Please, check that there are 14 pages in this booklet

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
Total	

(1) Consider the following system of linear equations,

$$\begin{aligned} x - y - bz =& 1 \\ x + (a-1)y =& b+1 \\ x + (a-1)y + (a-b)z =& 2b+1 \end{aligned}$$

where $a, b \in \mathbb{R}$ are parameters.

(a) Classify the system according to the values of the parameters a, b.

(b) Solve the above system for the values of a and b for which the system is underdetermined. How many parameters are needed to describe the solution?

(a) We compute first the ranks of the (augmented) matrix associated to the system. For this, we will do elementary operations.

$$(A|B) = \left(\begin{array}{ccc|c} 1 & -1 & -b & 1 \\ 1 & a-1 & 0 & 1+b \\ 1 & a-1 & a-b & 1+2b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -b & 1 \\ 0 & a & b & b \\ 0 & a & a & 2b \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & -b & 1 \\ 0 & a & b & b \\ 0 & 0 & a-b & b \end{array}\right)$$

We see that $\det A = a(a-b)$ is different from 0 when $a \neq 0$ and $a \neq b$. In this case $\operatorname{rank} A = \operatorname{rank}(A|B) = 3$ and the system has a unique solution.

If $a = 0$ ó $a = b$, then $\operatorname{rank} A \leq 2$. We study first the case $a = b$. If $a = b = 0$, then $\operatorname{rank} A = \operatorname{rank}(A|B) = 1$ and the system is underdetermined. If $a = b \neq 0$, then $\operatorname{rank} A = 2$ and $\operatorname{rank}(A|B) = 3$ so the system is overdetermined.

We study now the case $a \neq b$. If $a = 0 \neq b$, then $\operatorname{rank} A = 2$ and $\operatorname{rank}(A|B) = 3$ and the system is overdetermined.

Thus, the system

- has a unique solution if $a \neq 0$ and $a \neq b$. (since $\operatorname{rank} A = \operatorname{rank}(A|B) = 3$.)
- is underdetermined if $a = b = 0$. (since $\operatorname{rank} A = \operatorname{rank}(A|B) = 1$ and there are two parameters.)
- is overdetermined if $a = b \neq 0$. (since $\operatorname{rank} A = 2 \neq \operatorname{rank}(A|B) = 3$.)
- is overdetermined if $a = 0$ and $b \neq 0$. (since $\operatorname{rank} A = 2 \neq \operatorname{rank}(A|B) = 3$.)

(b) Plugging in the values $a = 0$ and $b = 0$, we see that the original system is equivalent to the following one,

$$x - y = 1$$

Taking y, z as parameters, we get $x = y + 1$, con $y, z \in \mathbb{R}$.

(2) The matrix

$$A = \begin{pmatrix} 7 & 1 & -1 \\ 2 & 8 & -2 \\ 1 & 1 & 5 \end{pmatrix}$$

has $\lambda_1 = 6$ and $\lambda_2 = 8$ as eigenvalues. (You do not need to prove this). Solve the following.

(a) Find the eigenvectors of the matrix A.

(b) Justify whether the matrix A is diagonalizable. And, if so, find two matrices D and P such that $A = PDP^{-1}$.

(c) Find a matrix B such that $B^2 = A$. It is enough to write B as the product of three matrices.

(a) The subspace of eigenvectors $S(6)$ is the set of solutions of the system

$$\begin{aligned} x + y - z &= 0 \\ 2x + 2y - 2z &= 0 \\ x + y - z &= 0 \end{aligned}$$

whose solution is $x = z - y$. Hence, $S(6) =< (1,0,1)(-1,1,0) >$.
The subspace of eigenvectors $S(8)$ is the set of solutions of the system

$$\begin{aligned} -x + y - z &= 0 \\ 2x - 2z &= 0 \\ x + y - 3z &= 0 \end{aligned}$$

whose solution is $x = z$, $y = 2z$. Hence, $S(8) =< (1,2,1) >$.

(b) In the previous part we have seen that $\dim S(6) = 2$ and $\dim S(8) = 1$. On the other hand, $\dim S(6) \le n_1$, $\dim S(8) \le n_2$ and $n_1 + n_2 = 3$. Thus, $2 \le n_1 \le 3$. That is, either $n_1 = 2$ or else $n_1 = 3$. But, $n_1 = 3$ is not compatible with $n_1 + n_2 = 3$ and $n_1 \ge 1$. We conclude that$n_1 = 2$ and $n_2 = 1$. And since $\dim S(6) = 2 = n_1$ and $\dim S(8) = 1 = n_2$, the matrix is diagonalizable.

Note: There are other ways to show that the matrix is diagonalizable. For example. (1) The set $\{(1,0,1)(-1,1,0),(1,2,1)\}$ is a basis of $\mathbb{R}^3$ which is formed by eigenvectors of A. Hence, A is diagonalizable. (2) The sum of the eigenvalues (counting their multiplicities) equals the trace of the matrix, that is 20. Thus, their are two alternatives $6+8+6 = 20$ ó $6+8+8 = 22$. Hence, the eigenvalues must be $\lambda_1 = 6$, $\lambda_2 = 8$, $\lambda_3 = 6$, so $n_1 = 2 = \dim S(6)$, $n_2 = 1 = \dim S(8)$ and the matrix is diagonalizable. (3) Compute the characteristic polynomial and check that it coincides with $(6-\lambda)^2(8-\lambda)$. Then, $n_1 = 2 = \dim S(6)$, $n_2 = 1 = \dim S(8)$ and the matrix is diagonalizable. (The last alternative is not very efficient, though).

The diagonal form, D and the matrix change of basis, P, are

$$D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

These matrices verify that $A = PDP^{-1}$.

(c) It is sufficient to take $B = PCP^{-1}$ where P is the matrix computed above and

$$C = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{8} \end{pmatrix}$$

Since $C^2 = D$, we have that $B^2 = \left(PCP^{-1}\right)\left(PCP^{-1}\right) = PC\left(P^{-1}P\right)CP^{-1} = PC^2P^{-1} = PDP^{-1} = A$.

(3) Given the linear map $f:\mathbb{R}^4\to\mathbb{R}^2$,

$$f(x,y,z)=(x-y+z-t,-x+y-z+t)$$

(a) Compute the dimensions of the kernel and the image and find some equations defining these subspaces.
(b) Find a basis of the image of f and a basis of the kernel of f.

(a) The matrix, A, of the function f, with respect to the the canonical bases is

$$A=\begin{pmatrix}1&-1&1&-1\\-1&1&-1&1\end{pmatrix}$$

We compute the reduced form,

$$\begin{pmatrix}1&-1&1&-1&x\\-1&1&-1&1&y\end{pmatrix}\sim\begin{pmatrix}1&-1&1&-1&x\\0&0&0&0&x+y\end{pmatrix}$$

from where we see that $\dim\operatorname{Im}(f)=\operatorname{rank}(A)=1$. A system of linear equations that define $\operatorname{Im}(f)$ is

$$x+y=0$$

From the formula

$$4=\dim(\operatorname{Im}(f))+\dim(\ker(f))$$

we see that $\dim(\ker(f))=3$ and A system of linear equations that define $\ker(f)$ is

$$x-y+z-t=0$$

(b) A basis of $\operatorname{Im}(f)$, obtained from the columns of A, is $\{(1,-1)\}$. We compute now a basis of $\ker(f)$. For this, we solve the above system,

$$x-y+z-t=0$$

We may take y,z,t as parameters and x as the dependent variable. Then,

$$\ker(f)=\{(x,y,z,t)\in\mathbb{R}^4:x=y-z+t\}=\{(y-z+t,y,z,t):y,z,t\in\mathbb{R}\}$$

and a basis of $\ker(f)$ is $\{(1,1,0,0),(-1,0,1,0),(1,0,0,1)\}$.

(4) Given the set

$$A = \{(x, y) \in \mathbb{R}^2 : \quad |y| \leq x, \quad x \leq 1\}$$

(a) Draw the set A, computing its vertices. Draw its boundary and interior and discuss whether the set A is open, closed, bounded, compact and/or convex. You must explain your answer.

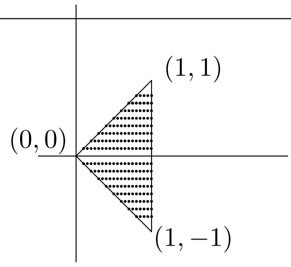
(b) Consider the function

$$f(x, y) = \frac{1}{(2x-1)^2 + (y-1)^2}$$

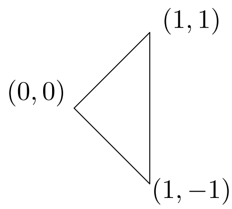
¿At what point(s) does the function f fail to be continuous? Determine if f attains a maximum and a minimum on the set A. State the theorems that you are using.

(c) Draw the level curves of the function $g(x, y) = (x-2)^2 + y^2$ and use them to determine the maxima and minima of g on A.

(a) The set A is



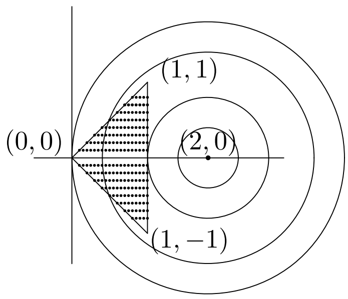
(b) The boundary (∂A) of A is



the interior of A is $A \setminus \partial A$, and the closure of A is $\bar{A} = A \cup \partial A = A$ (since $\partial A \subset A$). Therefore, A is closed, is not open (because $\partial A \cap A \neq \emptyset$), is compact (closed and bounded). Finally, the set A is convex. Another way to show that A is closed and convex is the following: The functions $h_1(x, y) = y - x$, $h_2(x) = y + x$ and $h_3(x, y) = 1 - x$ are continuous and linear. Hence, the set $A = \{(x, y) \in \mathbb{R}^2 : h_1(x, y) \leq 0, h_2(x, y) \geq 0, \quad h_3(x, y) \geq 0\}$ is closed and convex.

(b) The function f is continuous except at the point $(1/2, 1) \notin A$. Hence, f is continuous on A, which is compact. By Weierstrass' Theorem, the function attains a maximum and a minimum in the set A.

(c) The level curves of g satisfy the equation $(x-2)^2 + y^2 = C$, para $C \geq 0$. Hence, they are circumferences of center $(2, 0)$ and radius $\sqrt{C}$.



Graphically, we see that the (global) minimum is attained at the point $(1, 0)$ and que the (global) maximum is attained at the point $(0, 0)$.

(5) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Study if the function f is continuous at the point $(0,0)$. Study at which points of $\mathbb{R}^2$ the function f is continuous.
(b) Compute the partial derivatives of f at the point $(0,0)$.
(c) At which points of $\mathbb{R}^2$ is the function f differentiable?

(a) We study the limit when $(x,y) \to (0,0)$ by using straight lines $x(t) = t$, $y(t) = kt$ with $k \in \mathbb{R}$,

$$\lim_{t\to 0} f(t,kt) = \lim_{t\to 0} \frac{kt^2}{t^4 + k^2t^2} = \lim_{t\to 0} \frac{k}{t^2+k^2} = \frac{1}{k}$$

Since this limit depends on the parameter $k \in \mathbb{R}$, the limit does not exist and the function is not continuous at the point $(0,0)$.
Since the function is a quotient of polynomials and the denominator vanishes only at the point $(0,0)$, the function is continuous at every point $(x,y) \neq (0,0)$.

(b) The partial derivatives of f at the point $(0,0)$ are

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t}$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t}$$

We note that for every $t \neq 0$,

$$f(t,0) = \frac{0}{t^4} = 0$$
$$f(0,t) = \frac{0}{t^2} = 0$$

Therefore,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{0}{t} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{0}{t} = 0$$

(c) For $(x,y) \neq (0,0)$, the function $f(x,y)$ is defined as a quotient of polynomials and the denominator does not vanish. Hence, for $(x,y) \neq (0,0)$, all the partial derivatives exist and are continuous. We conclude that the function is differentiable at every points $(x,y) \neq (0,0)$.
En the point $(0,0)$ the function is not continuous y, hence, is not differentiable.

(6) Given the quadratic form

$$Q(x, y, z) = 2x^2 + y^2 + 3z^2 + 4axy + 2yz$$

(a) Determine the matrix associated to the above quadratic form.

(b) Classify the quadratic form, according to the values of the parameter a.

(a) The matrix associated to Q is

$$A = \begin{pmatrix} 2 & 2a & 0 \\ 2a & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

(b) We compute

$$D_1 = 2 > 0$$
$$D_2 = 2 - 4a^2$$
$$D_3 = 4 - 12a^2$$

We see that $D_2 \geq 0$ if and only if $2 - 4a^2 \geq 0$, that is if and only if

$$\frac{-1}{\sqrt{2}} \leq a \leq \frac{1}{\sqrt{2}}$$

On the other hand, $D_3 \geq 0$ if and only if $4 - 12a^2 \geq 0$, that is if and only if

$$\frac{-1}{\sqrt{3}} \leq a \leq \frac{1}{\sqrt{3}}$$

Hence, if

$$\frac{-1}{\sqrt{3}} < a < \frac{1}{\sqrt{3}}$$

then $D_1, D_2, D_3 > 0$ and Q is positive definite. If

$$a^2 = \frac{1}{3}$$

then $D_1, D_2 > 0, D_3 = 0$ and Q is positive semidefinite.
In all other cases, the quadratic form is indefinite.

(7) Consider the function
$$f(x,y) = x^2 - \ln(x^2) - 4\ln(y^2) + y^2$$

(a) Compute the gradient vector and the Hessian matrix of f at any point (x,y) in domain of the function.

(b) Determine the critical points of f and classify them.

(c) Determine if the function f attains any extreme points on the set
$$A = \{(x,y) \in \mathbb{R}^2 : x > 0, \quad y > 0\}$$

(a) We compute the partial derivatives
$$\nabla f(x,y) = (\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)) = (2x - \frac{2x}{x^2}, -4\frac{2y}{y^2} + 2y) = (2x - \frac{2}{x}, -\frac{8}{y} + 2y)$$
And the hessian matrix is
$$Hf(x,y) = \begin{pmatrix} 2 + \frac{2}{x^2} & 0 \\ 0 & \frac{8}{y^2} + 2 \end{pmatrix}$$

(b) The gradient exists at every point of the domain. Note that the partial derivatives of the function are continuous at every point of the domain. Thus, the function is differentiable at every point of the domain. Hence, the critical points satisfy the first order necessary condition $\nabla f(x,y) = (0,0)$. By using the expression of ∇f computed above and equating to 0 we get the following equations
$$\left.\begin{array}{rcl} 2(x^2-1) & = & 0 \\ 2(-4+y^2) & = & 0 \end{array}\right\}$$
solving for x in the first equation we obtain $x = 1, -1$. From the second one we get $y = 2, -2$. The critical points are $(1,2)$, $(1,-2)$, $(-1,2)$ and $(-1,-2)$; these are the only possible the critical points.
To classify them, we apply the second order sufficient condition. The hessian matrix is
$$Hf(1,2) = Hf(1,-2) = Hf(-1,2) = Hf(-1,2) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$
which is positive definite. Hence, the four critical points are local minima of f.

(c) The set A is the first quadrant of the plane and is an open set (since the inequalities are strict). Hence, we may not apply Weierstrass' Theorem.
On the other hand, the set A is convex (and open) and in A (actually in all of the domain of f) the Hessian matrix is positive definite. Therefore, the function is strictly convex in A. Therefore, at the point $(1,2)$ (which is a local minimum according the previous part) f attains the global minimum (in A). If f attained a global maximum in a point of A, this point would also be a local maximum. But, we have seen above that f does not have local maxima in A.

(8) Consider the function

$$f(x,y,z) = x + y - y^2 - x^2 - \frac{z^2}{2}$$

and the set

$$A = \{(x,y,z) : x + y + z = 0\}$$

(a) Find the Lagrange equations that determine the extreme points of f in the set A.
(b) Determine the points that satisfy the Lagrange equations and find the extreme points of f, specifying whether they correspond to a maximum or minimum.

(a) The Lagrangian function is

$$L(x,y,z,\lambda) = x + y - y^2 - x^2 - \frac{z^2}{2} + \lambda(x+y+z).$$

The Lagrange equations are:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 1 - 2x + \lambda = 0 \\
\frac{\partial L}{\partial y} &= 1 - 2y + \lambda = 0 \\
\frac{\partial L}{\partial z} &= -z + \lambda = 0 \\
& \quad x + y + z = 0
\end{aligned}$$

(b) From the first three equations, we obtain: $x = y$ and $z = 2x - 1$. Plugging these values in the las equation we get

$$x + x + 2x - 1 = 0$$

from where

$$x = \frac{1}{4}, \quad y = \frac{1}{4}, \quad z = -\frac{1}{2}, \quad \lambda = -\frac{1}{2}$$

We conclude that the unique point that satisfies the Lagrange equations is $\left(\frac{1}{4}, \frac{1}{4}, -\frac{1}{2}\right)$.
The Hessian matrix of the Lagrangian is

$$HL(x,y,z) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

which is clearly negative definite. Thus, $\left(\frac{1}{4}, \frac{1}{4}, -\frac{1}{2}\right)$ is a maximum of f in A. And there are no minima.