University Carlos III
Department of Economics
Mathematics II. Final Exam. May 2012

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.
- Read the exam carefully. Each part of the exam counts 1 point. Please, check that there are 10 pages in this booklet

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) *Consider the following system of linear equations*

$$\left\{\begin{array}{lcl} ax+y & = & 3 \\ x-az & = & 2 \\ y+z & = & b \end{array}\right.$$

where $a,b\in\mathbb{R}$.

(a) *Classify the system according to the values of* a *and* b.

(b) *Solve the above system for the values of* a *and* b *for which the system has infinitely many solutions.*

Solución:

(a) After some permutations of the equations, the matrices associated to the system are

$$A=\begin{pmatrix} 1 & 0 & -a \\ a & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$(A|B)=\left(\begin{array}{ccc|c} 1 & 0 & -a & 2 \\ 0 & 1 & 1 & b \\ a & 1 & 0 & 3 \end{array}\right)\mapsto\left(\begin{array}{ccc|c} 1 & 0 & -a & 2 \\ 0 & 1 & 1 & b \\ 0 & 1 & a^2 & 3-2a \end{array}\right)\mapsto\left(\begin{array}{ccc|c} 1 & 0 & -a & 2 \\ 0 & 1 & a^2 & 3-2a \\ 0 & 0 & 1-a^2 & b-3+2a \end{array}\right)=C$$

We study the possible ranks of A and compare them with the rank of $(A|B)$. We see that, if $a\neq 1$ and $a\neq -1$, then $\operatorname{rank}(A)=\operatorname{rank}(A|B)=3$. In any of these cases, the system is consistent and has a unique solution.
If $a=1$ then

$$C=\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & b-1 \end{array}\right)$$

and we see that if $b=1$, then $\operatorname{rank} C=2=\operatorname{rank} A$ and the system is underdetermined with one parameter. If $b\neq 1$, then $\operatorname{rank} C=3$, $\operatorname{rank} A=2$ and the system is inconsistent. Finally, if $a=-1$, then

$$C=\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & b-5 \end{array}\right)$$

and we see that if $b=5$, then $\operatorname{rank} C=2=\operatorname{rank} A$, so the system is underdetermined with one parameter. If $b\neq 5$, then $\operatorname{rank} C=3$, $\operatorname{rank} A=2$ and the system is inconsistent.

(b) The system is underdetermined when $a=1,b=1$ and when $a=-1,b=5$. For these values of a and b, the original system is equivalent to the following one

$$\left\{\begin{array}{lcr} x-az & = & 2 \\ y+z & = & 3-2a \end{array}\right.$$

whose solution is $x=2+az$, $y=3-2a-z$, $z\in\mathbb{R}$. Therefore, for $a=1,b=1$, the solution is

$$x=2+z,\quad y=1-z,\quad z\in\mathbb{R}$$

For $a=-1,b=5$, the solution is

$$x=2-z,\quad y=5-z,\quad z\in\mathbb{R}$$

(2) *Consider the set* $A = \{(x,y) \in \mathbb{R}^2 : y \geq -x^2 + 1, y \leq -x^2 + 4, x > 0, y \geq 0\}$ *and the function*

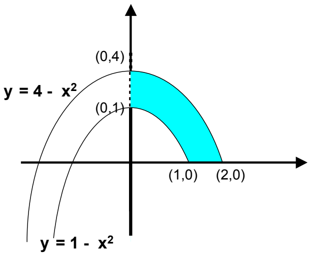
$$f(x,y) = x + \frac{y}{2}.$$

(a) *Draw the set* A*, its boundary and its interior. Determine, justifying your answers, whether the set* A *is closed, open, bounded, compact and/or convex.*

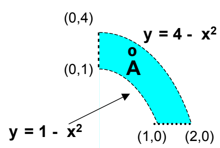
(b) *Are the hypotheses of Weierstrass' Theorem satisfied for the set* A *and the function* f*? Draw the level curves of* f *indicating the direction of growth. Use the level curves to determine (if they exist) the global maximum and/or minimum of* f *on* A *and the points at which they are attained.*

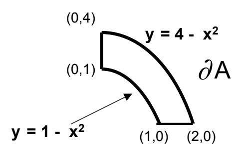
Solución:

(a) The representation of the set A is the following

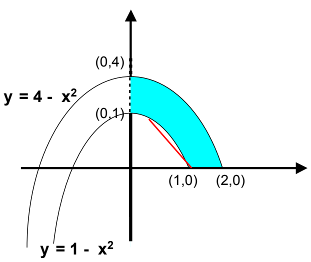


The interior and the boundary of the set A may be represented as



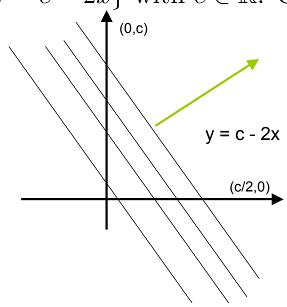


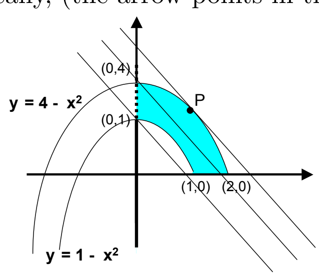
The set A is neither open (since, A does not coincide with its interior) nor closed (since, A does not contain its boundary). It is bounded, because it may be contained in the ball of center $(0,0)$ and radius 5. The set A is not convex because the line segment joining the points $(1,0)$ and $(1/2, 3/4)$ is not contained in A.



The set A is not compact, because it is not closed.

(b) The hypotheses of Weierstrass' Theorem are not satisfied because the set A is not compact (it is bounded but not closed). The level curves of f are the sets $\{(x,y) \in \mathbb{R}^2 : x + y/2 = c/2\} = \{(x,y) \in \mathbb{R}^2 : y = c - 2x\}$ with $c \in \mathbb{R}$. Graphically, (the arrow points in the direction of growth)





We see that the maximum value is attained at the point P at which the line $y = c - 2x$ is tangent to the graph of $y = 4 - x^2$. At that point, we have that $-2 = -2x$, that is, $x = 1$. And, from the equation $y = 4 - x^2$, we get that $P = (1,3)$. Therefore, the global maximum of f in A is $f(1,3) = 1 + 3/2 = 5/2$.

The minimum value of f in $\bar{A}$ (the closure of A) is attained at the point $(0,1) \notin A$. Graphically, we see that $f(x,y) > f(0,1) = 1/2$ for every $(x,y) \in A$. Since, $(0,1) \in \partial(A) \setminus A$, the function f takes on A any value arbitrarily close (but larger than) $f(0,1)$. We conclude that f does not attain a global minimum on A.

(3) *Answer the following questions.*

(a) *Given the function* $f(x,y) = y\ln xy - 3$, *compute the plane tangent to the graph of* f *corresponding to the point* $(x,y) = (1/2,2)$. *Compute the derivative of* f *at the point* $(1/2,2)$ *according to the vector* $v = (-1,3)$

(b) *Given the function* f *above, compute the Taylor polynomial of* f *of order two around the point* $(1/2,2)$.

Solución:

(a) The gradient vector is

$$\nabla f(x,y) = \left(\frac{y}{x}, \ln xy + 1\right)$$

At the point $(1/2,2)$ we have

$$\nabla f(1/2,2) = (4,1)$$

Since, $f(1/2,2) = -3$, the equation of the tangent plane is $4(x-1/2)+(y-2) = z+3$. That is, $4x+y-z = 7$. The derivative of f at the point $(1/2,2)$ according to the vector $v = (-1,3)$ is $\nabla f(1/2,2)\cdot v = (4,1)\cdot(-1,3) = -1$.

(b) The gradient associated to f is

$$\nabla f(1/2) = (4,1)$$

The Hessian matrix associated to f is

$$\mathrm{H}\, f(x,y) = \begin{pmatrix} -y/x^2 & 1/x \\ 1/x & 1/y \end{pmatrix}$$

At the point $(1/2,2)$ we obtain,

$$\mathrm{H}\, f(1/2,2) = \begin{pmatrix} -8 & 2 \\ 2 & 1/2 \end{pmatrix}$$

Taylor's second order polynomial is

$$P_2(x,y) = -3 + 4(x-1/2) + (y-2) - 4(x-1/2)^2 + \frac{1}{4}(y-2)^2 + 2(x-1/2)(y-2)$$

(4) *Consider the function* $f(x,y) = 8ax^3 - 24xy + y^3$, *where* $a \neq 0$.

(a) *Find the critical points of the function* f *above.*

(b) *Classify the critical points found above, according to the values of* a.

Solución:

(a) First, we compute the critical points of the function,

$$\frac{\partial f}{\partial x} = 24ax^2 - 24y = 0, \quad \frac{\partial f}{\partial y} = -24x + 3y^2 = 0$$

That is, $y = ax^2$, $y^2 = 8x$. The solutions are $x = y = 0$ and $x = 2a^{-2/3}$, $y = 4a^{-1/3}$.

(b) We compute the Hessian matrix associated to f:

$$Hf(x,y) = \begin{pmatrix} 48ax & -24 \\ -24 & 6y \end{pmatrix}$$

And we compute the Hessian matrix at the critical points,

$$Hf(0,0) = \begin{pmatrix} 0 & -24 \\ -24 & 0 \end{pmatrix}$$

Since, the determinant is $D_2 = -24^2 < 0$, the associated quadratic form is non-definite and $(0,0)$ is a saddle point. On the other hand,

$$Hf(2a^{-2/3}, 4a^{-1/3}) = \begin{pmatrix} 96a^{1/3} & -24 \\ -24 & 24a^{-1/3} \end{pmatrix} = 24 \begin{pmatrix} 4a^{1/3} & -1 \\ -1 & a^{-1/3} \end{pmatrix}$$

and we have that $D_1 = 96a^{1/3}$, $D_2 = 3 \times 24^2 = 1728 > 0$. Hence, if $a > 0$, the point $(2a^{-2/3}, 4a^{-1/3})$ is a local minimum. Whereas if $a < 0$ it is a local maximum.

(5) *Consider the function*

$$f(x, y) = x^4 - y^4$$

and the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

(a) *Find the Lagrange equations that determine the extreme points of f in A and calculate the solutions of these equations.*

(b) *Characterize the above solutions into local maxima and minima, using the second order conditions. Can you tell if they are global maxima and/or minima? (Explain your answer)*

Solución:

(a) Lagrange's function is

$$L(x, y, \lambda) = x^4 - y^4 - \lambda(x^2 + y^2 - 1)$$

The Lagrange's equations are

$$\begin{aligned} 4x^3 - 2\lambda x &= 0 \\ -4y^3 - 2\lambda y &= 0 \\ x^2 + y^2 &= 1 \end{aligned}$$

They may be simplified to

$$\begin{aligned} x(2x^2 - \lambda) &= 0 \quad (1) \\ y(2y^2 + \lambda) &= 0 \quad (2) \\ x^2 + y^2 &= 1 \quad (3) \end{aligned}$$

If $x = 0$, from the last equation we get that $y = \pm 1$. So, $\lambda = -2y^2 = -2$. Hence, the points

$$x = 0, y = \pm 1; \quad \lambda = -2$$

are solutions of Lagrange's equations. If $x \neq 0$, the first equations implies that $\lambda = 2x^2$. Substituting this value of λ into the second equation, we obtain that $0 = y(2y^2 + \lambda) = y(2y^2 + 2x^2) = 2y$. In the last step, we have used the third equation. Therefore, $y = 0$ and we see that the points

$$x = \pm 1, y = 0; \quad \lambda = 2$$

are solutions of the Lagrange equations, as well.

(b) The restriction is $g(x, y) = x^2 + y^2 - 1$ and we have that $\nabla g(x, y) = 2(x, y)$. The Hessian matrix associated to L is

$$\mathrm{H}\, L(x, y; \lambda) = \begin{pmatrix} 12x^2 - 2\lambda & 0 \\ 0 & -12y^2 - 2\lambda \end{pmatrix}$$

The associated quadratic form is $\mathrm{H}\, L$ es

$$Q(v_1, v_2) = (12x^2 - 2\lambda)v_1^2 - (12y^2 + 2\lambda)v_2^2$$

At the points $x = 0$, $y = \pm 1$, $\lambda = -2$, the associated vector subspace is $T = \{(v_1, v_2) \in \mathbb{R}^2 : (0, y) \cdot (v_1, v_2) = 0\} = \{(v_1, v_2) \in \mathbb{R}^2 : v_2 = 0\} = \{(x, y) \in \mathbb{R}^2 : v_2 = 0\}$, since, $y = \pm 1$. The associated quadratic form Q restricted to T is $Q^*(v_1) = 4v_1^2$, which is positive definite. We conclude that the points $(0, \pm 1)$ correspond to strict local minima of f.

At the pointss $x = \pm 1$, $y = 0$, $\lambda = 2$, the associated vector subspace is $T = \{(v_1, v_2) \in \mathbb{R}^2 : \nabla g(x, 0) \cdot (v_1, v_2) = 0\} = \{(v_1, v_2) \in \mathbb{R}^2 : xv_1 = 0\} = \{(x, y) \in \mathbb{R}^2 : v_1 = 0\}$, ya que $x = \pm 1$. The associated quadratic form Q restricted to T is $Q^*(v_2) = -4v_2^2$, which is negative definite. We conclude that the points $(\pm 1, 0)$ correspond to strict local maxima of f.

The set A is compact and the function f defined above is continuous. By Weierstrass' Theorem, the function f attains on A a maximum and a minimum value. Since, the regularity condition is satisfied, the points at which the extreme values are attained are solutions of of the Lagrange equations studied above. Since, $f(0, \pm 1) = -1$, $f(\pm 1, 0) = 1$, the function f attains in A its global minimum value at the points $(0, \pm 1)$. And the function f attains in A its global maximum value at the points $(\pm 1, 0)$.