(1) Given the following system of linear equations,

$$\left\{\begin{array}{rcl} ax+(3a+1)y+(3a+4)z & = & 1+a-b \\ x+2y+3z & = & 1 \\ -2ax+(2-2a)y+(7-5a)z & = & 3-2a-ab-2b \end{array}\right.$$

where $a, b \in \mathbb{R}$ are parameters.

(a) Classify the system according to the values of a and b. **1 point**

Solution: *The matrix associated with the system is*

$$\begin{pmatrix} a & 3a+1 & 3a+4 & a-b+1 \\ 1 & 2 & 3 & 1 \\ -2a & 2-2a & 7-5a & -ba-2a-2b+3 \end{pmatrix}$$

Exchanging rows 1 and 2 we obtain

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ a & 3a+1 & 3a+4 & a-b+1 \\ -2a & 2-2a & 7-5a & -ba-2a-2b+3 \end{pmatrix}$$

Next, we perform the following operations

$$\text{row } 2 \mapsto \text{row } 2 - a \times \text{row } 1$$

$$\text{row } 3 \mapsto \text{row } 3 + 2a \times \text{row } 1$$

And we obtain that the original system is equivalent to another one whose augmented matrix is the following

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & a+1 & 4 & 1-b \\ 0 & 2a+2 & a+7 & -ab-2b+3 \end{pmatrix}$$

Now, we perform the operation row 3 $\mapsto$ row 3 $- 2 \times$ row 2 and we obtain

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & a+1 & 4 & 1-b \\ 0 & 0 & a-1 & 1-ab \end{pmatrix}$$

Expanding the determinant using the last row, we obtain that the determinant of the system is $(a+1)(a-1)$. We conclude that if $a \neq 1$ and $a \neq -1$ then the system has a unique solution.

(i) *Suppose now that $a = 1$. The proposed system is equivalent to another one whose augmented matrix is*

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 1-b \\ 0 & 0 & 0 & 1-b \end{pmatrix}$$

(A) *If $b \neq 1$ the system has no solutions because* $\text{rank}(A) = 2 < \text{rank}(A|B) = 3$.

(B) *If $b = 1$ the system is undetermined with 1 parameter, since* $\text{rank}(A) = \text{rank}(A|B) = 2$.

(ii) *Suppose now that $a = -1$. The proposed system is equivalent to another one whose augmented matrix is*

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 4 & 1-b \\ 0 & 0 & -2 & b+1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & -2 & b+1 \\ 0 & 0 & 4 & 1-b \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & -2 & b+1 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$$

(A) *If $b \neq -3$ the system has no solutions because* $\text{rank}(A) = 2 < \text{rank}(A|B) = 3$.

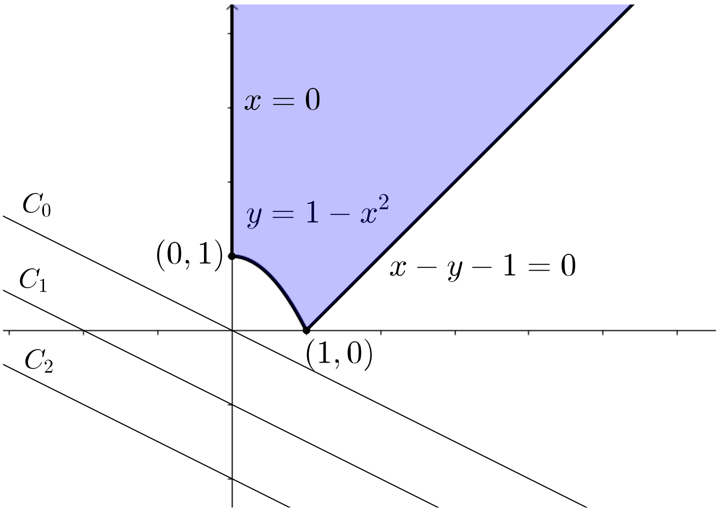
(B) *If $b = -3$ the systeme is underdetermined with 1 parameter, since* $\text{rank}(A) = \text{rank}(A|B) = 2$.

(b) Solve the above system for the values $a = -1$, $b = -3$. **1 point**

Solution: *The proposed system of linear equations is equivalente to the following one*

$$\left\{\begin{array}{rcl} x+2y+3z & = & 1 \\ -2z & = & -2 \end{array}\right.$$

Choosing y as the parameter, the set of solutions is $\{(-2-2y, y, 1) : y \in \mathbb{R}\}$.



(2) Consider the set $A = \{(x, y) \in \mathbb{R}^2 : 0 \leq x,\ 1 - x^2 \leq y,\ x - 1 \leq y\}$ and the function

$$f(x, y) = \frac{-x - 2y}{2}$$

defined on A.

(a) Sketch the graph of the set A and justify if it is open, closed, bounded, compact or convex. **1 point**

Solution: *The set A is approximately as indicated in the picture. It is closed because $\partial A \subset A$. It is not open because $A \cap \partial A \neq \emptyset$. It is not bounded, because no ball centered at the origin, contains the set A. Therefore, the set A is not compact. It is not convex, because the line segment that joins the points $(1, 0) \in A$ and $(0, 1) \in A$ is not contained in A.*

(b) Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A. Using the level curves, determine (if they exist) the extreme global points of f on the set A. **1 point**

Solution: *Weierstrass' Theorem does not apply because, even though the function f is continuous in all of $\mathbb{R}^2$, the set A is not compact, since it is not bounded.*
In the picture we represent three level curves. Note that following the level of growing level curves we reach a global maximun a the point $(1, 0)$. On the other hand, the function f does not have a global minimum on A. To see this, note that if we look at the points $(0, a) \in A$, when $a \to \infty$ we have that $f(0, a) = \frac{-2a}{2} \to -\infty$

(3) Consider the function $f(x,y) = x^3 + 2x^2 + 2xy - 16x + \frac{y^2}{2} - 2y - 4$.

(a) Determine the largest open subset of $\mathbb{R}^2$ where the function is strictly concave or convex. **1 point**

Solution: *The gradient of f is*

$$\left(3x^2 + 4x + 2y - 16, 2x + y - 2\right)$$

The Hessian matrix is

$$Hf(x,y) = \begin{pmatrix} 6x+4 & 2 \\ 2 & 1 \end{pmatrix}$$

We see that $D_1 = 6x + 4$ and $D_2 = 6x$. If $x > 0$, then $D_1 > 0$, $D_2 > 0$. We conclude that the function f is convex on the set $\{(x,y) \in \mathbb{R}^2 : x > 0\}$. The function is not strictly convex on any open set because if $D_1 < 0$ then $x < -2/3$ and we would have that $D_2 < 0$.

(b) Determine the critical points of the function f (if they exist) on $\mathbb{R}^2$. Classify the critical points of f on A. Determine if any of those critical points is a global extreme point. Justify your answer. **1 point**

Solution: *The equations defining the critical points are*

$$\begin{aligned} 0 &= 3x^2 + 4x + 2y - 16 \\ 0 &= 2x + y - 2 \end{aligned}$$

From the second equation we obtain that $y = 2 - 2x$. Substituting this value of y in the first equation, we obtain $3x^2 - 12 = 0$. That is, $x = \pm 2$. We conclude that the solutions are $(-2,6)$ and $(2,-2)$. Note that

$$H(2,-2) = \begin{pmatrix} 16 & 2 \\ 2 & 1 \end{pmatrix}$$

y

$$H(-2,6) = \begin{pmatrix} -8 & 2 \\ 2 & 1 \end{pmatrix}$$

so, $(2,-2)$ is a local minimum and $(-2,6)$ is a saddle point. Finally, note that $\lim_{y\to\infty} f(x,0) = +\infty$, $\lim_{x\to-\infty} f(x,0) = -\infty$ so there is neither a global maximum, nor a global minimum.

(4) Consider the set of equations

$$\begin{aligned} -u^3 + v^2 + x^2 - y^2 + 4 &= 0 \\ -2u^2 + 3v^4 + 2xy + y^2 + 8 &= 0 \end{aligned}$$

(a) Prove that the above system of equations determines implicitly two differentiable functions $u(x,y)$ and $v(x,y)$ in a neighborhood of the point $(x,y,u,v) = (2,-1,2,1)$. **0,5 points**

Solution: *Let* $f_1(x,y,u,v) = -u^3+v^2+x^2-y^2+4$, $f_2(x,y,u,v) = -2u^2+3v^4+2xy+y^2+8$. *These functions are differentiable of any order. Further,* $f_1(2,-1,2,1) = f_2(2,-1,2,1) = 0$. *We compute*

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{vmatrix} = 8uv - 36u^2v^3$$

which at the point $(x,y,u,v) = (2,-1,2,1)$ *takes the value* -128. *We have checked that the assumptions of the implicit function theorem hold. Therefore the equations* $f_1(x,y,u,v) = 0$, $f_2(x,y,u,v) = 0$ *define implicitly differentiable functions* $u(x,y)$ *and* $v(x,y)$ *in a neigborhood of the point* $(x,y,u,v) = (2,-1,2,1)$.

(b) Compute

$$\frac{\partial u}{\partial x}(2,-1), \quad \frac{\partial v}{\partial x}(2,-1) \quad \frac{\partial u}{\partial y}(2,-1), \quad \frac{\partial v}{\partial y}(2,-1)$$

1 point

Solution: *Differentiating implicitly with respect to* x,

$$\begin{aligned} 2x - 3u^2\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} &= 0 \\ 2y - 4u\frac{\partial u}{\partial x} + 12v^3\frac{\partial v}{\partial x} &= 0 \end{aligned}$$

we plug in the values $(x,y,u,v) = (2,-1,2,1)$ *to obtain the following*

$$\begin{aligned} 4 - 12\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial x} &= 0 \\ -2 - 8\frac{\partial u}{\partial x} + 12\frac{\partial v}{\partial x} &= 0 \end{aligned}$$

Hence, $\frac{\partial u}{\partial x}(2,-1) = \frac{13}{32}$, $\frac{\partial v}{\partial x}(2,-1) = \frac{7}{16}$. *Differentiating implicitly with respect to* y,

$$\begin{aligned} -2y - 3u^2\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} &= 0 \\ 2x + 2y - 4u\frac{\partial u}{\partial y} + 12v^3\frac{\partial v}{\partial y} &= 0 \end{aligned}$$

we plug in the values $(x,y,u,v) = (2,-1,2,1)$ *to obtain the following*

$$\begin{aligned} 2 - 12\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} &= 0 \\ 2 - 8\frac{\partial u}{\partial y} + 12\frac{\partial v}{\partial y} &= 0 \end{aligned}$$

Hence, $\frac{\partial u}{\partial y}(2,-1) = \frac{5}{32}$, $\frac{\partial v}{\partial y}(2,-1) = -\frac{1}{16}$.

(c) Using the previous part and Taylor's polynomial of order 1 of the function $u(x,y)$, compute approximately the value of $u(1.99,-1.019)$. **0,5 points**

Solution: *Recall Taylor's polynomial of order 1 of* $u(x,y)$ *at the point* (a,b)

$$P_2(x,y) = u(a,b) + \frac{\partial u}{\partial x}(a,b)(x-a) + \frac{\partial u}{\partial y}(a,b)(y-b)$$

we plug in the values $(a,b) = (2,-1)$, $(x,y) = (1.99,-1.019)$,

$$P_2(1.99,-1.019) = u(2,-1) + \frac{\partial u}{\partial x}(2,-1)(-0.01) + \frac{\partial u}{\partial y}(2,-1)(-0.019) = 2 - 0.01 \times \frac{13}{32} - 0.019 \times \frac{5}{32} = 1.93203$$

(5) Consider the function $f(x, y, z) = x^2 + y^2 + z^2 - 3x - 4y$ and the sphere of equation $x^2 + y^2 + z^2 = 25$.

(a) Check that the hypotheses of Lagrange's Theorem hold. Write the Lagrange equations for f on the sphere. compute the points that satisfy those equations and the values of the associated Lagrange multipliers. **1 point**

(b) Assuming that the sphere is closed and bounded and using part (a) above, determine the extreme points of the function f on the sphere. Determine which of those points correspond to global maxima or minima. Justify your answer. **1 point**

Solution:

(a) *The objective function f and the restriction $h(x, y, z) = x^2 + y^2 + z^2 - 25$ are both of class C^1 (in fact, they are of class C^n for any n). In addition, the gradient of h, $\nabla h(x, y, z) = (2x, 2y, 2z)$, vanishes only at $(0, 0, 0)$, which is not feasible. Hence, the assumptions of the Lagrange Theorem are fulfilled. The extreme points of f on the sphere are critical points of the Lagrangian*

$$L(z, y, z, \lambda) = x^2 + y^2 + z^2 - 3x - 4y - \lambda(x^2 + y^2 + z^2 - 25).$$

The lagrange equations are:

$$\begin{cases} \frac{\partial L}{\partial x}(x, y, z) & = & 2x - 3 - 2x\lambda = 0 \\ \frac{\partial L}{\partial y}(x, y, z) & = & 2y - 4 - 2y\lambda = 0 \\ \frac{\partial L}{\partial z}(x, y, z) & = & 2z - 2z\lambda = 0 \\ \frac{\partial L}{\partial \lambda}(x, y, z) & = & -(x^2 + y^2 + z^2 - 25) = 0. \end{cases}$$

The third equation can be written as $2z(1 - \lambda) = 0$. Note that $\lambda = 1$ is in contradiction with the first and the second equation, hence $z = 0$. From the the first and second equations we obtain

$$x = \frac{3}{2(1 - \lambda)}, \quad y = \frac{4}{2(1 - \lambda)}$$

Plugging these values for x, y and $z = 0$ into the equation of the sphere, we have

$$\frac{9}{4(1 - \lambda)^2} + \frac{16}{4(1 - \lambda)^2} = 25,$$

and solving for $(1 - \lambda)^2$, we find $(1 - \lambda)^2 = \frac{25}{100} = \frac{1}{4}$, that is, $1 - \lambda = \pm\frac{1}{2}$. Plugging these two values of $1 - \lambda$ into the expression of x and y above, we find two critical points

$$P_1 = (3, 4, 0), \text{ with } \lambda_1 = \frac{1}{2} \text{ and } P_2 = (-3, -4, 0) \text{ with} \lambda_2 = \frac{3}{2}.$$

(b) *We use the second order conditions to classify critical points. The Hessian matrix of the Lagrangian with respect to (x, y, z) is*

$$\mathcal{H}L_{x,y,z}(x, y, z) = \begin{pmatrix} 2(1 - \lambda) & 0 & 0 \\ 0 & 2(1 - \lambda) & 0 \\ 0 & 0 & 2(1 - \lambda) \end{pmatrix}.$$

At the point P_1, the Hessian matrix is positive definite, thus P_1 is a local minimum of f on the sphere. At the point P_2, the Hessian matrix is negative definite, thus P_2 is a local maximum of f on the sphere.

Alternatively, we can apply Weierstrass' Theorem , since the sphere is a compact set and the objective function is continuous. Thus, f admits global extrema on the sphere. Since these extreme points satisfy the Lagrange equations, we conclude that P_1 corresponds to a global minimum and P_2 corresponds to a global maximum of f on the sphere.