(1) Consider the following system of linear equations with a parameter $a \in \mathbb{R}$.

$$\begin{cases} ax + y + z &= b \\ ax + ay + z &= a \\ x + ay + az &= 1 \end{cases}$$

Please, answer the following questions.

(a) Classify the system according to the values of a. **1 point**

Solution: *The augmented matrix is*

$$\begin{pmatrix} a & 1 & 1 & b \\ a & a & 1 & a \\ 1 & a & a & 1 \end{pmatrix}$$

After elementary row operations we obtain

$$\begin{pmatrix} 1 & a & a & 1 \\ a & a & 1 & a \\ a & 1 & 1 & b \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & a & 1 \\ 0 & a-a^2 & 1-a^2 & 0 \\ 0 & 1-a^2 & 1-a^2 & b-a \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & a & 1 \\ 0 & a-a^2 & 1-a^2 & 0 \\ 0 & 1-a & 0 & b-a \end{pmatrix}$$

We see that the original system is equivalent to another system of linear equations whose associated matrix is

$$\begin{pmatrix} 1 & a & a & 1 \\ 0 & a-a^2 & 1-a^2 & 0 \\ 0 & 1-a & 0 & b-a \end{pmatrix}$$

Expanding the determinant using the last row, we see that the determinant associated to the system is $(1-a)(1-a^2) = (1-a)^2(1+a)$. *We conclude that if* $a \neq 1$ *and* $a \neq -1$ *then, the system is consistent.*

Suppose now that $a = 1$. *The original system is equivalent to another system of linear equations whose associated matrix is*

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b-1 \end{pmatrix}$$

If $b \neq 1$ *the system is inconsistent. Whereas, if* $b = 1$, *the system is underdetermined with* $3-1=2$ *parameters.*

Suppose now that $a = -1$. *The original system is equivalent to another system of linear equations whose associated matrix is*

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & b+1 \end{pmatrix}$$

And we see that $\operatorname{rank}(A) = 2$. *If* $b \neq -1$, *then* $\operatorname{rank}(A|b) = 3$ *the system is inconsistent. Whereas, if* $b = -1$, *then* $\operatorname{rank}(A|b) = 2$ *the system is underdetermined with* $3-2=1$ *parameters.*

(b) Solve the above system for the values $a = b = -1$. **1 point**

Solution: *The original system is equivalent to the following onw*

$$\begin{cases} x - y - z &= 1 \\ 2y &= 0 \end{cases}$$

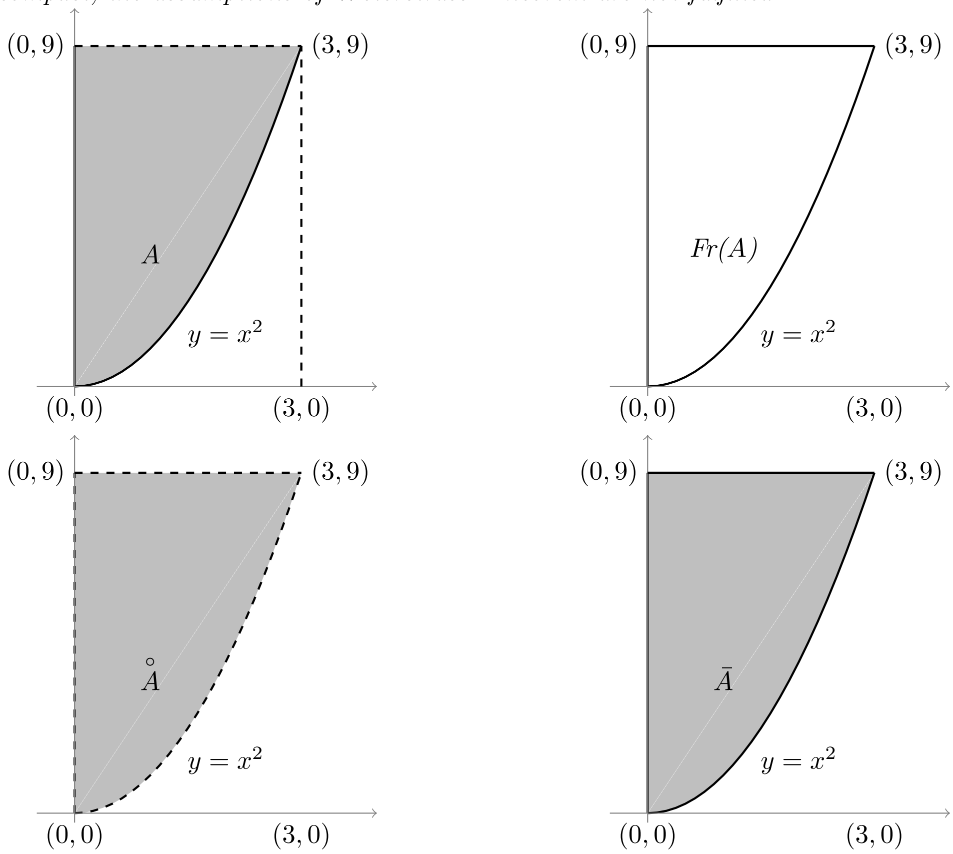
Choosing z *as a parameter, the set of solutions is* $\{(1+z, 0, z) : z \in \mathbb{R}\}$.

(2) Consider the function $f(x, y) = 4x - y$ and the set

$$A = \{(x, y) \in \mathbb{R}^2 \, : \, 0 \leq x \leq 3, \; 0 \leq y < 9, \; x^2 \leq y\}.$$

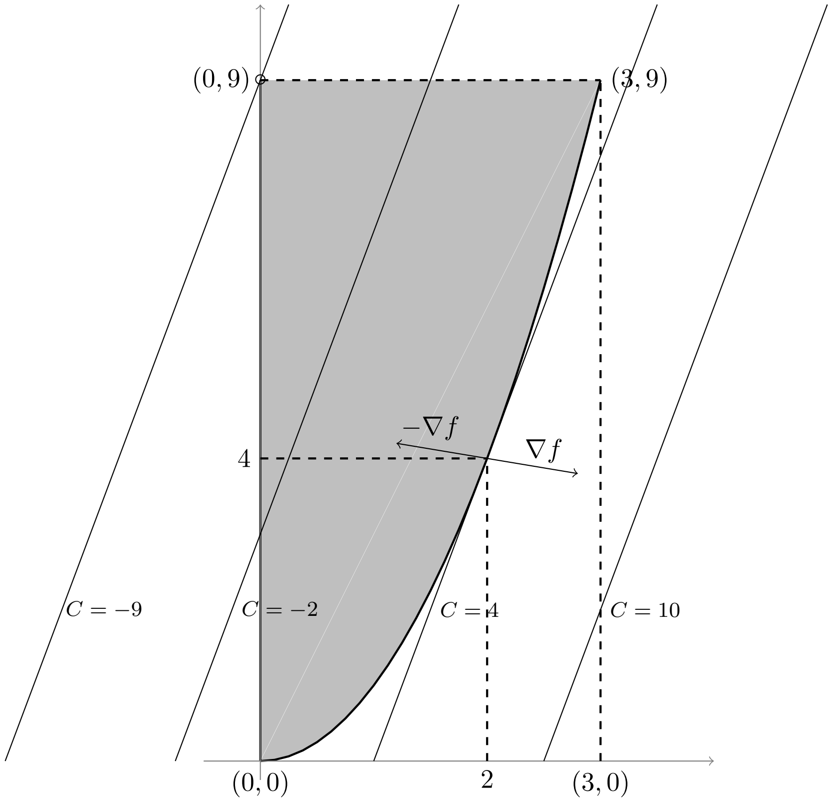
(a) Represent the set A, its boundary, closure and interior. Argue whether the function f and the et A satisfy the conditions of Weierstrass' Theorem. **1 point**

Solution: *The set A is not closed: it does not contain its boundary because the line segment joining the points $(0, 9)$ and $(3, 9)$ is contained in the boundary of A, but not in A. Since A is not compact, the assumptions of Weierstrass' Theorem are not fulfilled.*



(b) Represent the level curves of the function f on the set A, indicating the directions in which f increases/decreases. Using the level curves, determine (if they exist) the global extreme points of f on A. **1 point**

Solution: *The level curves are straight lines with slope 4, $4x - y = C$, where $C \in \mathbb{R}$. The gradient of f, $\nabla f = (4, -1)$, points in the direction of maximal growth of f, and $-\nabla f = (-4, 1)$ points in the direction in which f decreases the fastest. The maximum value of f on A is attained at the point (a, b) at which the level curve of f is tangent to the graph of $y = x^2$. The slope of the straight line tangent to the graph of $y = x^2$ at the point (a, b) is $2a$. Hence, $2a = 4$, that is $a = 2$. On the other hand, $b = a^2 = 4$ and $4a - b = C$ implies $C = 4$. We conclude that f attains its maximum value on A at the point $(a, b) = (2, 4)$ and the maximum value is $C = 4$. To discuss the existence of minima, note that the function f decreases as its level curves move towards the upper left corner of the set A. Thus, if the point $(0, 9)$ would belong to A, f would attain its minimum value on A at that point. However, since $(0, 9)$ does not belong to A, the function f does not attain a global*



minimum value on A.

(3) Consider the function $f(x, y) = bx^2 + y^3 - 6bxy$ with $b \in \mathbb{R}$, $b \neq 0$.

(a) Determine the critical points (if they exist) of the function f on the set $\mathbb{R}^2$. **1 point**

Solution: *The gradient of f is*

$$\left(2bx - 6by, 3y^2 - 6bx\right)$$

The critical points are determined by the following equations

$$\begin{aligned} 0 &= b(2x - 6y) \\ 0 &= 3y^2 - 6bx \end{aligned}$$

Since, $b \neq 0$, the solutions are $(0, 0)$, $(18b, 6b)$.

(b) Classify the critical points found above into (local or global) maximum, minimum and saddle points. **1 point**

Solution: *The Hessian matrix is*

$$H(x, y) = \begin{pmatrix} 2b & -6b \\ -6b & 6y \end{pmatrix}$$

We see that

$$H(0, 0) = \begin{pmatrix} 2b & -6b \\ -6b & 0 \end{pmatrix}$$

Since, $\det(H(0, 0)) = -36b^2 < 0$, the point $(0, 0)$ is a saddle point. On the other hand,

$$H(18b, 6b) = \begin{pmatrix} 2b & -6b \\ -6b & 36b \end{pmatrix}$$

We see that $D_1 = 2b$ and $D_2 = 36b^2 > 0$. We conclude that if $b > 0$, the point $(18b, 6b)$ corresponds to a local minimum, whereas if $b < 0$ the point $(18b, 6b)$ corresponds to a local maximum.
Finally, $f(0, y) = y^3$ and we see that $\lim_{y \to \infty} f(0, y) = +\infty$, $\lim_{y \to -\infty} f(0, y) = -\infty$ so there are no global maximum or minimum points.

(4) Consider the equation $3xz - 8y^3 - z^3 + 6z = 3$.

(a) Prove that the above equation defines a differentiable function $z(x, y)$ in a neighbourhood of the point $(2, 1, 1)$. **1 point**

Solution: *Let* $f(x, y, z) = 3xz - 8y^3 - z^3 + 6z - 3$. *Since,*

$$\frac{\partial f}{\partial z}(2, 1, 1) = 3x - 3z^2 + 6\big|_{x=2, y=1, z=1} = 9$$

by the implicit function Theorem, the equation $3xz - 8y^3 - z^3 + 6z = 3$ *defines* z *as a differentiable function of the variables* x *and* y, *in a neighbourhood of the point* $(2, 1)$.

(b) Compute Taylor's polinomial of order 1 of the function $z(x, y)$, computed above, at the point $(2, 1)$. **1 point**

Solution: *Differentiating implicitly the equation* $3xz - 8y^3 - z^3 + 6z = 3$ *with respect to the variables* x *and* y *we obtain*

$$\begin{aligned} 3z + 3x\frac{\partial z}{\partial x} - 3z^2\frac{\partial z}{\partial x} + 6\frac{\partial z}{\partial x} &= 0 \\ 3x\frac{\partial z}{\partial y} - 24y^2 - 3z^2\frac{\partial z}{\partial y} + 6\frac{\partial z}{\partial y} &= 0 \end{aligned}$$

and substituting now $x = 2, y = 1, z = 1$ *we get the equations*

$$\begin{aligned} 3 + 9\frac{\partial z}{\partial x}(2, 1) &= 0 \\ 9\frac{\partial z}{\partial y}(2, 1) - 24 &= 0 \end{aligned}$$

from these we obtain

$$\frac{\partial z}{\partial x}(2, 1) = \frac{-1}{3}, \quad \frac{\partial z}{\partial y}(2, 1) = \frac{8}{3}$$

Therefore, Taylor's polinomial of order 1 of the function $z(x, y)$ *at the point* $(2, 1)$ *is*

$$P_1(x, y) = 1 - \frac{x - 2}{3} + \frac{8}{3}(y - 1)$$

(5) Consider the function $f(x,y,z) = 2x + y^2 + z^2$ on the set $A = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 9, z = 0\}$

(a) Write the Lagrange equations for f on the set A. Compute the points that satisfy those equations and the value of the corresponding Lagrange multipliers. **1 point**

Solution: *The Lagrange function of the problem is* $L(x,y,z;\lambda,\mu) = 2x + y^2 + z^2 + \lambda(x^2 + y^2 + z^2) + \mu z$. *The Lagrange equations are*

$$\begin{cases} \frac{\partial L}{\partial x} & = 2 + 2\lambda x = 0, \\ \frac{\partial L}{\partial y} & = 2y + 2\lambda y = 0, \\ \frac{\partial L}{\partial z} & = 2z + 2\lambda z + \mu = 0, \\ \frac{\partial L}{\partial \lambda} & = x^2 + y^2 + z^2 - 9 = 0 \\ \frac{\partial L}{\partial \mu} & = z = 0 \end{cases} .$$

From the fifth equation we obtain $z = 0$ *and from the third* $\mu = 0$. *Therefore the above system reduces to*

$$\begin{cases} \frac{\partial L}{\partial x} & = 2 + 2\lambda x = 0, \\ \frac{\partial L}{\partial y} & = 2y + 2\lambda y = 0, \\ \frac{\partial L}{\partial \lambda} & = x^2 + y^2 - 9 = 0 \end{cases} .$$

From the second equation we obtain $2y(1+\lambda) = 0$. *Therefore,*

- *either* $y = 0$ *and substituting in the fourth equation* $x^2 + 0 + 0 = 9$ *we obtain* $x = \pm 3$, *and we get the solutions* $(3,0,0;-\frac{1}{3},0)$ *and* $(-3,0,0;\frac{1}{3},0)$.
- *or* $\lambda = -1$ *and substituting in the first equation we get* $x = 1$ *and substituting now in the fourth equation* $1 + y^2 + 0 = 9$ *we obtain* $y = \pm\sqrt{8}$, *and we get the solutions* $(1,\sqrt{8},0;-1,0)$ *y* $(1,-\sqrt{8},0;-1,0)$.

(b) Knowing that the set A is closed and bounded, study the existence of global extreme points of f on A and compute those points. **1 point**

Solution: *Since,* A *is a compact set and* f *being a polynomial is continuous, Weierstrass's Theorem guarantees that* f *attains a global maximum and minimum on* A. *These points satisfy the Lagrange equations computed above. We compute the value of the function at those points* $f(3,0,0) = 6$, $f(-3,0,0) = -6$ $f(1,\sqrt{8},0) = 10$ *y* $f(1,-\sqrt{8},0) = 10$ *and we see that* f *attains the minimum value at the point* $(-3,0,0)$ *and the maximum value at the third and fourth points.*