University Carlos III
Department of Economics
Mathematics II. Final Exam. May 2016

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(1) Consider the following system of linear equations with parameters $a, b \in \mathbb{R}$

$$\begin{cases} ax + y + z & = & b \\ x + ay + z & = & 1 \\ x + (2a-1)y + z & = & 2-b \end{cases}$$

(a) Classify the system according to the values of a and b. **1 point**

Solution: *Using Gauss' method, the row elementary operations, we have obtained the following augmented matrix of a linear system of equations equivalent to the given system:*

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1-a & 0 & -1+b \\ 0 & 0 & 1-a & 1-ab \end{pmatrix}$$

Firstly, if $a \neq 1$ the system is consistent and determined, with only one solution. Secondly, supposing that $a = 1$ then the system has the following equivalent augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1+b \\ 0 & 0 & 0 & 1-b \end{pmatrix}$$

If $b \neq 1$ then the system is inconsistent and if $b = 1$ the system is consistent and underdetermined depending on two parameters.

(b) Solve the above system of linear equations for the values of the parameters $a = b = 1$. **1 point**
Solution: *In that case the given linear system is equivalent to:*

$$x + y + z = 1$$

If we choose x, y as the free parameters the set of solutions is $\{(x, y, 1-x-y) : x, y \in \mathbb{R}\}$.

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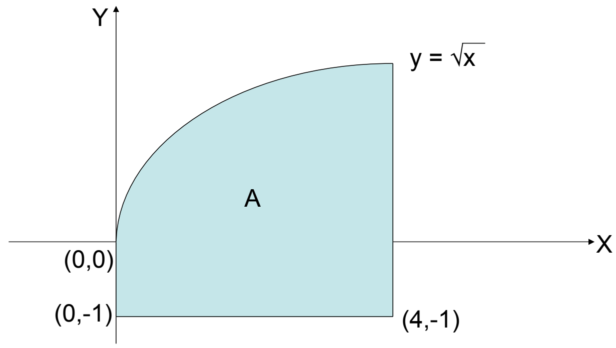
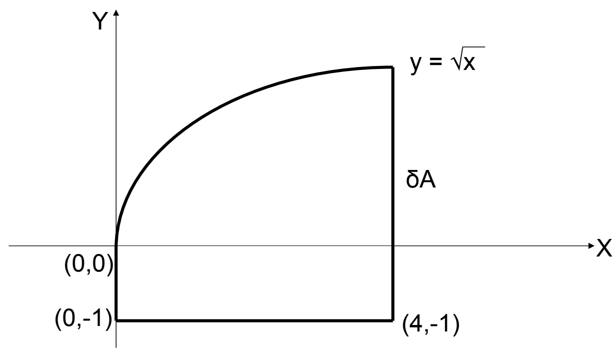
(2) Consider the function $f(x, y) = x - y$ and the set $A = \{(x, y) \in \mathbb{R}^2 : -1 \leq y \leq \sqrt{x}, 0 \leq x \leq 4\}$.

(a) Represent graphically the set A, its boundary, closure and interior. Justify if the function f attains a global maximum and/or a minimum in the set A, without computing them. $\boxed{\textbf{1 point}}$

Solution: *We can draw the set A as the set of points on the plane:*



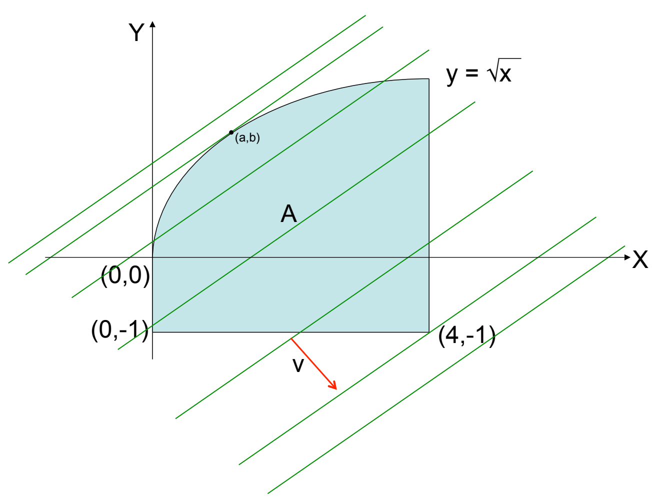
Its boundary.



Because the boundary is included in the set A, it's closed. A is also bounded, therefore compact. Since the function f is continuous, using Weierstrass' theorem, we know that f attains its global maximum and minimum values on the set A.

(b) Using the level curves of the function f, compute the global extreme points of the previous part. $\boxed{\textbf{1 point}}$

Solution: *We can sketch the level curves of f as the graphs of the functions $y = x - C$ that depend on $C \in \mathbb{R}$.*



The vector v aims towards the growing direction of f. We can notice that the maximum value is 5 and it is attained at the point $(4, -1)$. The minimum value is attained at the tangent point (a, b) between the graph of $f(x) = \sqrt{x}$ and the straight line $g(x) = x - C$ for some value of $C \in \mathbb{R}$. In that case, it must verified $f'(a) = g'(a)$, i.e.

$$\frac{1}{2\sqrt{a}} = 1$$

then $a = 1/4$, $b = f(a) = 1/2$. The global minimum value of f on A is $a - b = -1/4$.

(3) Consider the function

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}e^{xy} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

(a) Compute the partial derivatives of f at the point $(0,0)$. $\boxed{\textbf{1 point}}$

Solution: *The partial derivatives of f at the point $(0,0)$ are*

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=\lim_{t\to 0}\frac{0\cdot t\cdot e^0}{t(0+t^2)}==\lim_{t\to 0}\frac{0}{t^3}=0\\ \frac{\partial f}{\partial x}(0,0) &= \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t}=\lim_{t\to 0}\frac{t\cdot 0\cdot e^0}{t(t^2+0)}=\lim_{t\to 0}\frac{0}{t^3}=0\end{aligned}$$

(b) Determine if f is continuous and/or differentiable at the point $(0,0)$. $\boxed{\textbf{1 point}}$

Solution: *If we try the curve $\alpha(t)=(t,t)$ we obtain*

$$\lim_{t\to 0}f(\alpha(t))=\lim_{t\to 0}\frac{t^2}{2t^2}e^{t^2}=\frac{1}{2}$$

Meanwhile, if we use the curve $\sigma(t)=(t,t^2)$ we can see that

$$\lim_{t\to 0}f(\sigma(t))=\lim_{t\to 0}\frac{t^3}{t^2+t^4}e^{t^3}=\lim_{t\to 0}\frac{t}{1+t}e^{t^3}=0$$

so the limit doesn't exist nor is the function continuous at the point $(0,0)$. Since, the function is not continuous at $(0,0)$ it isn't differentiable either.

(4) An agent has a utility function $f(x,y,z) = \ln x + 2\ln y + 3\ln z$. The agent chooses the bundle $(x,y,z) \in \mathbb{R}^3$ which maximizes his utility function $f(x,y,z)$ on his budget line

$$A = \{(x,y,z) \in \mathbb{R}^3 : x + 2y + 3z = 90, x > 0, y > 0, z > 0\}$$

(a) Write the Lagrange equations satisfied by the extreme points of f on the set A. Compute the points which satisfy the Lagrange equations and the value of the Lagrange multiplier, corresponding to each of those points. **1 point**

Solution: *The lagrangian function for the problem is* $L = \ln x + 2\ln y + 3\ln z + \lambda\,(90 - x - 2y - 3z)$. *Then the lagrangian equations are:*

$$\frac{1}{x} - l = 0, \frac{2}{y} - 2l = 0, \frac{3}{z} - 3l = 0, x + 2y + 3z = 90$$

whose solutions are $x = y = z = l = 1/15$.

(b) Classify the solutions of the previous part into local maxima and minima, using the second order conditions. Can we say if any of the previous local maxima and minima is a global maximum or minimum in the set A? Justify the answers. **1 point**

Solution: *The hessian matrix is*

$$\begin{pmatrix} -\frac{1}{x^2} & 0 & 0 \\ 0 & -\frac{2}{y^2} & 0 \\ 0 & 0 & -\frac{3}{z^2} \end{pmatrix}$$

is clearly negative definite on the set A. *The set of points* A *is open and convex, the function* f *is convex on* A *because its hessian matrix is negative definite on that set. Thus, the point obtained in the previous part is the global maximum of* f *on* A.

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(5) Consider the function $f(x, y) = 10 - 8x + 2x^2 + 8y - 4xy + bx^2y + 2y^2$ con $b > 2$.

(a) Determine the critical points of the function f on the set $\mathbb{R}^2$. $\boxed{\textbf{1 point}}$

Solution: *The gradient vector of f is*

$$\left(2bxy + 4x - 4y - 8, bx^2 - 4x + 4y + 8\right)$$

then, the equations to calculate the critical points are:

$$\begin{aligned} 0 &= 2bxy + 4x - 4y - 8 \\ 0 &= bx^2 - 4x + 4y + 8 \end{aligned}$$

Adding both equations we get $bx(x + 2y) = 0$. One solution is $x = 0$, $y = -2$. If $x \neq 0$, then $x = -2y$. Putting that value back in the second equation we obtain the new equation $4by^2 + 12y + 8 = 0$ whose discriminant is $9 - 8b < 0$ thus, the equation has no real solutions. The only solution is $x = 0$, $y = -2$.

(b) Classify the critical points of f into (local or global) maxima, minima and saddle points, depending on the values of the parameter b. $\boxed{\textbf{1 point}}$

Solution: *The hessian matrix is*

$$\begin{pmatrix} 2by + 4 & 2bx - 4 \\ 2bx - 4 & 4 \end{pmatrix}$$

and if we evaluate it at the point $x = 0$, $y = -2$ is

$$H = \begin{pmatrix} 4 - 4b & -4 \\ -4 & 4 \end{pmatrix}$$

Now, looking at the leading principals $D_1 = 4 - 4b < 0$, $D_2 = -16b < 0$, we can see that H is indefinite and the critical point $x = 0$, $y = -2$ is a saddle point. There aren't any local or global extreme values.