$\boxed{1}$

Given the parameter $a \in \mathbb{R}$, consider the following linear system.

$$\left\{\begin{array}{ccccccc} x & +3y & & -2t & = & -1 \\ 2x & +y & +z & +t & = & -2 \\ 3x & -y & +z & -t & = & 7 \\ 2x & +6y & +z & +t & = & a \end{array}\right.$$

(a) (6 points) Discuss the system depending on the values of the parameter a.

(b) (4 points) Solve the system whenever it is possible.

Solutions:

(a) We use Gausian elimination to transform the augmented matrix of the system

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & -2 & -1 \\ 2 & 1 & 1 & 1 & -2 \\ 3 & -1 & 1 & -1 & 7 \\ 2 & 6 & 1 & 1 & a \end{array}\right),$$

into an equivalent echelon matrix.

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & -2 & -1 \\ 2 & 1 & 1 & 1 & -2 \\ 3 & -1 & 1 & -1 & 7 \\ 2 & 6 & 1 & 1 & a \end{array}\right) \underset{\sim}{\substack{r_2-2r_1 \\ r_3-3r_1 \\ r_4-2r_1}} \left(\begin{array}{cccc|c} 1 & 3 & 0 & -2 & -1 \\ 0 & -5 & 1 & 5 & 0 \\ 0 & -10 & 1 & 5 & 10 \\ 0 & 0 & 1 & 5 & a+2 \end{array}\right)$$

$$\underset{\sim}{r_3-2r_2} \left(\begin{array}{cccc|c} 1 & 3 & 0 & -2 & -1 \\ 0 & -5 & 1 & 5 & 0 \\ 0 & 0 & -1 & -5 & 10 \\ 0 & 0 & 1 & 5 & a+2 \end{array}\right) \underset{\sim}{r_4+r_3} \left(\begin{array}{cccc|c} 1 & 3 & 0 & -2 & -1 \\ 0 & -5 & 1 & 5 & 0 \\ 0 & 0 & -1 & -5 & 10 \\ 0 & 0 & 0 & 0 & a+12 \end{array}\right).$$

Two cases appear:

- $a = -12$. The bottom line is null and the rank of the matrix of the system and of the augmented matrix coincides and equals 3, thus the system admits infinitely many solutions.
- $a \neq -12$. The bottom line is not null and the rank of the matrix of the system is 3, smaller than the rank of the augmented matrix, thus the system has no solutions.

(b) The equivalent echelon system can be rewritten

$$\left\{\begin{array}{ccccc} x & +3y & & = & -1-2t \\ & -5y & +z & = & -5-t \\ & & -z & = & 10+5t \end{array}\right. .$$

After taking the parameter $t = \lambda$, the solutions are $(x = 5 + 2\lambda, y = -2, z = -10 - 5\lambda, t = \lambda)$.

$\boxed{2}$

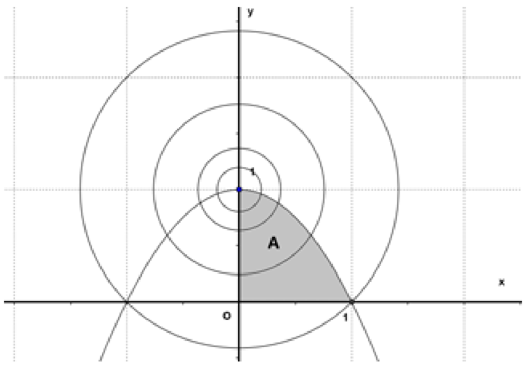
Consider the set $A = \{(x, y) \in \mathbb{R}^2 : y \leq 1 - x^2, x \geq 0, y \geq 0\}$ and the function $f(x, y) = x^2 + (y - 1)^2$.

(a) (5 points) Draw the set A and prove that the function f attains global maximum and global minimum value in the set A.

(b) (5 points) Draw the level curves of f. Using this information, find the points where the global maximum and global minimum value of f are obtained in A, and the value of f in those points.

Solutions:

(a) The set A is the shadow region represented in the figure below.



The set A is closed, as it contains its boundary points. It is also bounded, as can be enclosed by a finite ball, for instance the ball of center $(0, 0)$ and radius 2. in cosnequence, A is compact. Function f is continuous in $\mathbb{R}^2$ because it is a polynomial. Hence f is also continuous in A. By the Theorem of Weierstrass, the function f attains global extrema (maximum and minimum) in A.

(b) The level curves of f are given by $x^2 + (y - 1)^2 = k$, with $k \geq 0$. They are circumferences with center in $(0, 1)$ and radius $\sqrt{k}$. Obviously, the level increases with the radius of the circumferences. Thus, the global minimum of f in A is $(0, 1)$, with value $f(0, 1) = 0$. The global maximum of f in A is $(1, 0)$, with value $f(1, 0) = 2$.

$\boxed{3}$

Given the parameter $a \in \mathbb{R}$ with $a \neq 0$, consider the function $f(x,y) = x^4 - 2ax^2 - y^2 + 3$.

(a) (4 points) Calculate the critical points of f depending on the value of $a \neq 0$.

(b) (6 points) Classify the critical points of f into local maxima, minima and saddle points, depending on the value of $a \neq 0$.

Solutions:

(a) The function f is of class C^2 in $\mathbb{R}^2$ since it is a polynomial. The gradient of f is

$$\nabla f(x,y) = (4x^3 - 4ax, -2y).$$

The critical points are the solutions of the system of equations $4x^3 - 4ax = 0$ and $-2y = 0$. The solutions are $(0,0)$ independently of the value of a and the additional points $(\pm\sqrt{a}, 0)$ when $a > 0$. In summary:

– If $a > 0$, then the critical points are $(0,0)$, $(\sqrt{a}, 0)$ and $(-\sqrt{a}, 0)$.

– If $a < 0$, then $(0,0)$ is the only critical point.

(b) We will study the character of the critical points through the sign of the Hessian matrix of f, which is given by

$$\mathcal{H}f(x,y) = \begin{pmatrix} 12x^2 - 4a & 0 \\ 0 & -2 \end{pmatrix}.$$

Note that the matrix is diagonal. This fact makes quite easy to study the sign of the quadratic form.

<u>Case i) $a > 0$.</u> We have three points to consider. The Hessian matrices are

$$\mathcal{H}f(0,0) = \begin{pmatrix} -4a & 0 \\ 0 & -2 \end{pmatrix}, \quad \mathcal{H}f(\sqrt{a},0) = \begin{pmatrix} 8a & 0 \\ 0 & -2 \end{pmatrix}, \quad \mathcal{H}f(-\sqrt{a},0) = \begin{pmatrix} 8a & 0 \\ 0 & -2 \end{pmatrix}.$$

The first matrix is negative definite because $a > 0$, hence $(0,0)$ a (strict) local maximum of f. in the two remaining cases, the matrix is indefinite, thus both critical points , $(\pm\sqrt{a}, 0)$, are saddle points of f.

<u>Cas2 ii) $a < 0$.</u> The Hessian matrix is as before

$$\mathcal{H}f(0,0) = \begin{pmatrix} -4a & 0 \\ 0 & -2 \end{pmatrix},$$

but now it is indefinite, as $a < 0$. In consequence $(0,0)$ is now a saddle point of f.

$\boxed{4}$

Consider a firm that produces a good by using labor and capital. The production function of the firm is $F(x,y) = 60x^{\frac{1}{3}}y^{\frac{2}{3}}$, where x and y represent the cost of labor and capital (in thousands of euros), respectively.

(a) (4 points) Study whether the function F is concave or convex in the set

$$A = \{(x,y) \in \mathbb{R}^2 \,:\, x > 0, y > 0\}.$$

(b) (6 points) Suppose that the budget of the firm is 120 (thousands of euros) and the firm's objective is to produce the greatest possible amount of good. Write the optimization problem of the firm and solve the Lagrange equations.

Solutions:

(a) It is clear that A is a convex set since it is intersection of two open semiplanes, which are convex sets. The function F is of class C^2 in A. The gradient of F is $(20x^{-\frac{2}{3}}y^{\frac{2}{3}}, 40x^{\frac{1}{3}}y^{-\frac{1}{3}})$ and the Hessian matrix is thus

$$\mathcal{H}F(x,y) = \begin{pmatrix} -\frac{40}{3}x^{-\frac{5}{3}}y^{\frac{2}{3}} & \frac{40}{3}x^{-\frac{2}{3}}y^{-\frac{1}{3}} \\ \frac{40}{3}x^{-\frac{2}{3}}y^{-\frac{1}{3}} & -\frac{40}{3}x^{\frac{1}{3}}y^{-\frac{4}{3}} \end{pmatrix}.$$

The first principal minor is negative for all $(x,y) \in A$, $-\frac{40}{3}x^{-\frac{5}{3}}y^{\frac{2}{3}} < 0$. The second principal minor is the determinant, that equals

$$\left(\frac{40}{3}\right)^2 x^{-\frac{4}{3}}y^{-\frac{2}{3}} - \left(\frac{40}{3}\right)^2 x^{-\frac{4}{3}}y^{-\frac{2}{3}} = 0.$$

Thus the Hesian matrix is negative semidefinite for all $(x,y) \in A$, hence F is concave.

(b) The Lagrangian of the problem is

$$\mathcal{L}(x,y,\lambda) = 60x^{\frac{1}{3}}y^{\frac{2}{3}} + \lambda(120 - x - y).$$

The critical points of $\mathcal{L}$ are the solutions of the system $\nabla\mathcal{L} = \overline{0}$, that is,

$$\left\{\begin{array}{rcl} 20x^{-\frac{2}{3}}y^{\frac{2}{3}} - \lambda & = & 0 \\ 40x^{\frac{1}{3}}y^{-\frac{1}{3}} - \lambda & = & 0 \\ x + y & = & 120 \end{array}\right.$$

After eliminating λ from the two first equations we find the relationship $x^{-\frac{2}{3}}y^{\frac{2}{3}} = 2x^{\frac{1}{3}}y^{-\frac{1}{3}}$, or $y = 2x$. By plugging it into the third equations we get the point $(x^*, y^*) = (40, 80)$ with multiplier $\lambda^* = 20\sqrt[3]{4}$.

$\boxed{5}$

Consider the equation $x^2y + e^{zx} = 1$.

(a) (4 points) Indicate in which of the following points $p = (0, 1, 1)$ and $q = (1, 0, 0)$ the Implicit Function Theorem can be applied, so that the equation above defines z as a function of x and y in a neighborhood of each point.

(b) (6 points) Calculate $\dfrac{\partial z}{\partial x}(x, y)$ and $\dfrac{\partial z}{\partial y}(x, y)$ at those points of part (a) where the hypotheses of the Implicit Function Theorem were fulfilled.

Solutions:

(a) Let $f(x, y, z) = x^2y + e^{zx} - 1$ be the function that defines the equation.

- The function f is formed by elementary functions (polynomial and exponential), thus it is of class C^1.
- It is easy to check that both points, p and q satisfy the equation.
- The determinant of the matrix of partial derivatives with respect to z is $\frac{\partial f}{\partial z}(x, y) = e^{zx}x$, a 1×1 matrix in this case. The value at p is 0, thus the hypothesis of the theorem are not fulfilled for p. The value at q is 1, different from 0, thus the hypotheses of the theorem are fulfilled for q.

(b) By part (a) we focus on q.

Deriving with respect to x in the equation we get

$$2xy + e^{zx}(\frac{\partial z}{\partial x}(x, y)x + z) = 0$$

and substituting $q = (1, 0, 0)$

$$2 \cdot 1 \cdot 0 + e^{0 \cdot 1}(\frac{\partial z}{\partial x}(1, 0) \cdot 1 + 0) = 0.$$

Finally, $\frac{\partial z}{\partial x}(1, 0) = 0$

Deriving with respect to y in the equation we get

$$x^2 + e^{zx}(\frac{\partial z}{\partial y}(x, y)x) = 0.$$

and substituting $q = (1, 0, 0)$

$$1^2 + e^{0 \cdot 1}(\frac{\partial z}{\partial y}(1, 0) \cdot 1) = 0.$$

Finally, $\frac{\partial z}{\partial y}(1, 0) = -1$.

$\boxed{6}$

Consider the function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) (5 points) Study the continuity of f in $(0,0)$.

(b) (5 points) Calculate the partial derivatives

$$\frac{\partial f}{\partial x}(0,0), \quad \frac{\partial f}{\partial y}(0,0)$$

and the directional derivative of f at $(0,0)$ along the direction $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. In other words, compute $D_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})}f(0,0)$.

Solutions:

(a) The limits through lines $y = mx$ passing through $(0,0)$ are all equal to 0. Now, if we compute the limit through the parabola $y = x^2$, then the limit is $\frac{1}{2} \neq 0$, thus the limit does not exist and then f is not continuous at $(0,0)$. Here is the computation:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2x^2}{x^4+(x^2)^2} = \frac{1}{2},$$

(b) It is clear that both partial derivatives exist and are zero at $(0,0)$, since $f(h,0) = f(0,k) = f(0,0) = 0$. To find the directional derivative at $(0,0)$, we compute the following limit:

$$\lim_{h\to 0} \frac{f(\frac{h}{\sqrt{2}}, \frac{h}{\sqrt{2}}) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\frac{h^2}{2}\frac{h}{\sqrt{2}}}{\frac{h^4}{4}+\frac{h^2}{2}}}{h} = \lim_{h\to 0} \frac{\sqrt{2}h^3}{h^5+2h^3} = \lim_{h\to 0} \frac{\sqrt{2}}{h^2+2} = \frac{\sqrt{2}}{2}.$$

Hence $D_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})}f(0,0) = \frac{\sqrt{2}}{2}$.