(1) *Consider the following system of linear equations*

$$\left\{\begin{array}{rcr} 3x+2y+2az & = & -13 \\ x+2y+az & = & -5 \\ 5x+2ay-3z & = & -18 \end{array}\right.$$

where $a \in \mathbb{R}$ is a constant.

(a) *Classify the system according to the values of a.*

(b) *Solve the above linear system for the values of a for which the system has infinitely many solutions. How many parameters are needed to describe the solution?*

Solución:

(a) The augmented matrix of the system is the following

$$(A|B) = \left(\begin{array}{ccc|c} 3 & 2 & 2a & -13 \\ 1 & 2 & a & -5 \\ 5 & 2a & -3 & -18 \end{array}\right)$$

Performing elementary row operations we get that

$$(f_1 \rightleftharpoons f_2) \left(\begin{array}{ccc|c} 1 & 2 & a & -5 \\ 3 & 2 & 2a & -13 \\ 5 & 2a & -3 & -18 \end{array}\right) \quad (f_2 \mapsto 3f_1 - f2) \left(\begin{array}{ccc|c} 1 & 2 & a & -5 \\ 0 & 4 & a & -2 \\ 5 & 2a & -3 & -18 \end{array}\right) \quad (f_3 \mapsto 5f_1 - f3) \left(\begin{array}{ccc|c} 1 & 2 & a & -5 \\ 0 & 4 & a & -2 \\ 0 & 10-2a & 5a+3 & -7 \end{array}\right)$$

$$(f_3 \mapsto (10-2a)f_2/4 - f3) \left(\begin{array}{ccc|c} 1 & 2 & a & -5 \\ 0 & 4 & a & -2 \\ 0 & 0 & -\frac{a^2}{2} - \frac{5a}{2} - 3 & a+2 \end{array}\right) \quad (f_3 \mapsto 2f3) \left(\begin{array}{ccc|c} 1 & 2 & a & -5 \\ 0 & 4 & a & -2 \\ 0 & 0 & -a^2-5a-6 & 2a+4 \end{array}\right)$$

(We may also compute directly the determinant $|A| = -2(a^2+5a+6)$.)

The rank of A is 3 if $-a^2-5a-6 \neq 0$ and 2 if $-a^2-5a-6=0$. Therefore, the rank of A is 2 if and only if $a=-2$ ó $a=-3$.

- If $a=-2$, then $\text{rank}(A) = \text{rank}(A|B) = 2$ and the system is under-determined.
- If $a=-3$, then $\text{rank}(A) = 2 < \text{rank}(A|B) = 3$ and the system is inconsistent.
- If $a \neq -2$ and $a \neq -3$, then $\text{rank}(A) = \text{rank}(A|B) = 3$ and the system is consistent.

(b) The system is under-determined if $a=-2$. In this case, the original system is equivalent to the following linear system

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & -5 \\ 0 & 2 & -1 & -1 \end{array}\right)$$

Taking y as the parameter the set of solutions is

$$\{(2y-3, y, 2y+1) : z \in \mathbb{R}\}$$

(2) *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$.

(a) *Write the definition that* f *is continuous at the point* $p = (x_0, y_0)$.

(b) *Consider the function*

$$f(x,y) = \begin{cases} \frac{2x^2y}{x^4+y^2} & \textit{if } (x,y) \neq (0,0), \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$$

Determine if the function f *is continuous at the point* $(0,0)$.

Solución:

(a) The function f is continuous at the point $p = (x_0, y_0)$ if

$$\lim_{(x,y)\to p} f(x,y) = f(p)$$

That is, f is continuous at the point $p = (x_0, y_0)$ if given $\varepsilon > 0$, there is a $\delta > 0$ such that if

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

then $|f(x,y) - f(p)| < \varepsilon$

(b) Taking a curve of the form $y = x^2$ we see that

$$\lim_{x\to 0} f(x, y^2) = \lim_{x\to 0} \frac{2x^4}{x^4 + x^4} = 1 \neq f(0,0)$$

so, the function f is not continuous at the point $(0,0)$.

(3) *Consider the function*

$$f(x,y) = 2xe^{y/x},$$

the point $p = (2,0)$ *and the vector* $v = (3,2)$.

(a) *Compute the gradient of the function* f *at the point* p. *Compute the tangent plane to the graph of* f *at the point* $(2,0,4)$.

(b) *Compute the directional derivative of the function* f *at the point* p *in the direction of the vector* v. *Which is the direction of maximal growth of* f *at the point* p? *Which is the maximum value of the directional derivative?*

Solución:

(a) The function is differentiable at the points $(x,y) \in \mathbb{R}^2$ such that $x \neq 0$. The gradient of f at a point $(x,y) \in \mathbb{R}^2$ such that $x \neq 0$ is

$$\nabla f(x,y) = \left(2e^{y/x} - \frac{2y}{x}e^{y/x}, 2e^{y/x}\right) = e^{y/x}\left(2 - \frac{2y}{x}, 2\right)$$

which evaluated at the point $p = (2,0)$ is $\nabla f(2,0) = (2,2)$. Now, we note that $f(2,0) = 4$, so the point $(2,0,4)$ is on the graph of f. The tangent plane to the graph of f at the point $(2,0,4)$ is determined by the equation

$$\nabla f(2,0) \cdot (x-2, y) = z - 4$$

that is, $z = 2x + 2y$.

(b) The derivative of the function f at the point p according to the vector v is

$$D_v f(p) = \nabla f(p) \cdot v = (2,2) \cdot (3,2) = 10$$

Since $\|v\| = \sqrt{13}$, the directional derivative of the function f at the point p in the direction the vector v is

$$\frac{1}{\|v\|} D_v f(p) = \frac{10}{\sqrt{13}}$$

The direction of maximal growth of f at the point p in the direction determined by $\nabla f(p)$ is

$$\frac{1}{\|\nabla f(p)\|}\nabla f(p) = \frac{1}{2\sqrt{2}}(2,2) = \frac{1}{\sqrt{2}}(1,1)$$

and the maximum value of the function's directional derivative at the point p is

$$\|\nabla f(p)\| = \|(2,2)\| = 2\sqrt{2}$$

(4) *Consider the function*

$$f(x, y, z) = x^2 - y^2 + 2xy - \frac{z^2}{2} + 6$$

and the set

$$A = \{(x, y, z) \in \mathbb{R}^3 : z = 2x + 2y - 1\}$$

(a) *Compute the Lagrange equations that determine the extreme values of f on A.*
(b) *Determine the points that satisfy the Lagrange equations.*

Solución:

(a) The Lagrangian associated to the problem is

$$L(x, y, z) = x^2 - y^2 + 2xy - \frac{z^2}{2} + 6 + \lambda(z - 2x - 2y + 1)$$

The Lagrange equations are

$$\begin{aligned} 2x + 2y - 2\lambda &= 0 \\ -2y + 2x - 2\lambda &= 0 \\ -z + \lambda &= 0 \\ 2x + 2y - 1 &= z \end{aligned}$$

that is,

$$\begin{aligned} x + y &= \lambda \\ x - y &= \lambda \\ z &= \lambda \\ 2x + 2y - 1 &= z \end{aligned}$$

(b) From the first two equations we obtain $x + y = x - y$ which implies that $y = 0$. Therefore, the equations can be reduced to

$$\begin{aligned} x &= \lambda \\ z &= \lambda \\ 2x - 1 &= z \end{aligned}$$

We obtain $\lambda = z = 2\lambda - 1$ so $\lambda = x = 1$. The solution is $x = 1$, $y = 0$, $z = 1$.

(5) *Consider the following maximization problem*

$$\max_{x,y} \quad y(x-1)$$
$$\textbf{s.t.} \quad (x-1)^2 + y^2 \le 2$$

(a) *Compute the Kuhn-Tucker equations that determine the extreme values of f on A.*
(b) *Determine the points that satisfy the Kuhn-Tucker equations.*

Solución:

(a) The function $f(x,y) = y(x-1)$ is continuous and the set $\{(x-1)^2 + y^2 \le 2\}$ is compact. By Weierstrass' Theorem there is a global maximum and a global minimum. The functions $y(x-1)$ and $(x-1)^2 + y^2 - 2$ are of class C^∞ and the regularity conditions hold. The Lagrangian del problem is

$$L = y(x-1) + \lambda(2 - (x-1)^2 - y^2)$$

and the Kuhn-Tucker equations are

$$\frac{\partial L}{\partial x} = y - 2\lambda(x-1) = 0$$
$$\frac{\partial L}{\partial y} = x - 1 - 2\lambda y = 0$$
$$\lambda(2 - (x-1)^2 - y^2) = 0$$
$$(x-1)^2 + y^2 \le 2$$
$$\lambda \ge 0$$

(b) With $\lambda = 0$, we obtain the solution $x = 1$, $y = 0$. Let us look for solutions with $\lambda \neq 0$. Assuming this, the Kuhn-Tucker equations are

(1) $$y = 2\lambda(x-1)$$
(2) $$x - 1 = 2\lambda y$$
(3) $$(x-1)^2 + y^2 = 2$$
(4) $$\lambda > 0$$

Substituting $x - 1 = 2\lambda y$ in the equation (1), we obtain that $4\lambda^2 y = y$. A possible solution is $y = 0$ and we obtain again $x = 1$. But, this is not compatible with $(x-1)^2 + y^2 = 2$. We conclude that $y \neq 0$ and, hence

$$\lambda^2 = \frac{1}{4}$$

Since, $\lambda > 0$ we obtain

$$\lambda = \frac{1}{2}$$

We may the write the Kuhn-Tucker equations as

$$y = x - 1$$
$$(x-1)^2 + y^2 = 2$$
$$\lambda = 1/2$$

Substituting $y = x - 1$ in the second equation, we obtain $2y^2 = 2$, that is, $y = \pm 1$. Therefore, we obtain the the solutions

$$x = 2, y = 1, \lambda = \frac{1}{2}, \qquad x = 0, y = -1, \lambda = \frac{1}{2}$$

Since, $f(1,0) = 0$, $f(2,1) = 1 = f(0,-1)$ the maximum value is attained at the points $(2,1)$ and $(0,-1)$.