University Carlos III
Department of Economics
Mathematics II. Final Exam. June 2018

(1) Given the following system of linear equations,

$$\left\{ \begin{array}{rcl} x + ay & = & b \\ ax + z & = & b \\ ay + z & = & 2 \end{array} \right.$$

where $a, b \in \mathbb{R}$ are parameters.

(a) Classify the system according to the values of a and b. **1 point**

Solution: *The matrix associated with the system is*

$$\begin{pmatrix} 1 & a & 0 & b \\ a & 0 & 1 & b \\ 0 & a & 1 & 2 \end{pmatrix}$$

Exchanging rows 2 and 3 we obtain

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & a & 1 & 2 \\ a & 0 & 1 & b \end{pmatrix}$$

Next, we perform the operation row 3 $\mapsto$ *row 3* $- a \times$ *row 1. We obtain that the original system is equivalent to another one whose augmented matrix is the following*

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & a & 1 & 2 \\ 0 & -a^2 & 1 & b - ab \end{pmatrix}$$

Now, we perform the operation row 3 $\mapsto$ *row 3* $+ a \times$ *row 2 and we obtain*

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & a & 1 & 2 \\ 0 & 0 & 1 + a & b - ab + 2a \end{pmatrix}$$

Expanding the determinant using the last row, we obtain that the determinant of the system is $a(a+1)$. *We conclude that if* $a \neq 0$ *and* $a \neq -1$ *then the system has a unique solution,*

$$z = \frac{b - ab + 2a}{1 + a}, \quad y = 2 - \frac{b - ab + 2a}{a(1 + a)}, \quad x = b - 2a + \frac{b - ab + 2a}{1 + a}$$

(i) *Suppose now that* $a = 0$. *The proposed system is equivalent to another one whose augmented matrix is*

$$\begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & b \end{pmatrix}$$

(A) *If* $b \neq 2$ *the system has no solutions because* $rank(A) = 2 < rank(A|B) = 3$.
(B) *If* $b = 2$ *the system is undetermined with* 1 *parameter, since* $rank(A) = rank(A|B) = 2$.

(ii) *Suppose now that* $a = -1$. *The proposed system is equivalent to another one whose augmented matrix is*

$$\begin{pmatrix} 1 & -1 & 0 & b \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 2b - 2 \end{pmatrix}$$

(A) *If* $b \neq 1$ *the system has no solutions because* $rank(A) = 2 < rank(A|B) = 3$.
(B) *If* $b = 1$ *the system is underdetermined with* 1 *parameter, since* $rank(A) = rank(A|B) = 2$.

(b) Solve the above system for the values $a = -1$, $b = 1$. **1 point**

Solution: *The proposed system of linear equations is equivalente to the following one*

$$\left\{ \begin{array}{rcl} x - y & = & 1 \\ -y + z & = & 2 \end{array} \right.$$

Choosing y *as the parameter, the set of solutions is* $\{(1+y, y, y+2) : y \in \mathbb{R}\}$.

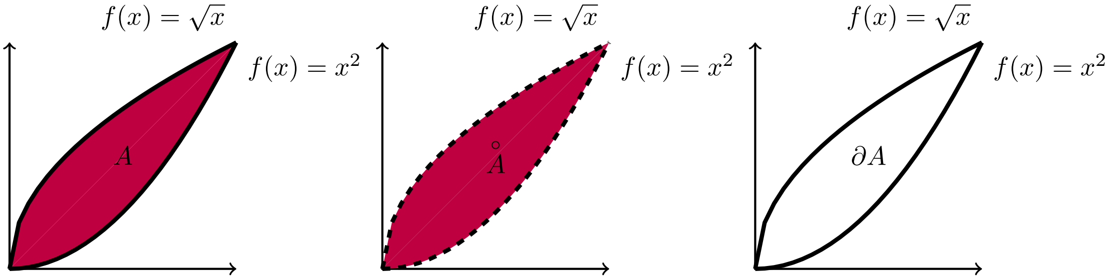
(2) Consider the set $A = \{(x, y) \in \mathbb{R}^2 : \quad 0 \leq x, \quad x^2 \leq y \leq \sqrt{x}\}$ and the function

$$f(x, y) = x + y$$

defined on A.

(a) Sketch the graph of the set A and justify if it is open, closed, bounded, compact or convex. **1 point**

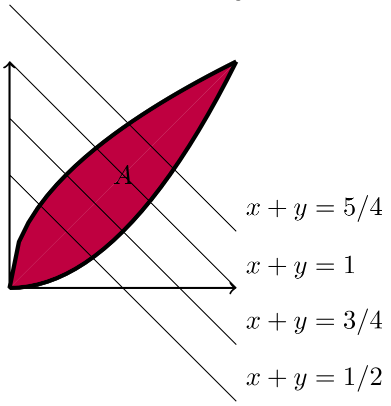
Solution: *The set A is approximately as indicated in the picture.*



It is closed because $\partial A \subset A$. It is not open because $A \cap \partial A \neq \emptyset$. It is bounded, because the ball centered at the origin with radius 2 contains the set A. Therefore, the set A is compact. It is convex.

(b) Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A. Using the level curves, determine (if they exist) the extreme global points of f on the set A. **1 point**

Solution: *Weierstrass' Theorem may be applied because, the function f is continuous in all of $\mathbb{R}^2$ and the set A is compact.*
In the picture we represent three level curves. Note that following the level of growing level curves we reach a global maximun a the point $(1, 1)$ and a global minimum at the point $(0, 0)$.



(3) Consider the function $f(x,y) = x^2 e^y$.

(a) Compute the equation of the plane tangent to the graph of the function f at the point $(-1,0,1)$. **1 point**

Solution: *The gradient vector of* f *is* $\nabla f(x,y) = \left(2xe^y, x^2e^y\right)$. *Computing it at the point* $(-1,0)$ *we obtain* $\nabla f(-1,0) = (-2,1)$. *The equation of the tangent plane is* $-2(x+1)+y-(z-1)=0$. *That is,*

$$z = 1 - 2(x+1) + y$$

(b) Compute Taylor's polynomial of degree 2 of the function f at the point $(-1,0)$. **1 point**

Solution: *The Hessian matrix is*

$$Hf(x,y) = \begin{pmatrix} 2e^y & 2xe^y \\ 2xe^y & x^2e^y \end{pmatrix}$$

which at the point $(-1,0)$ *becomes*

$$Hf(-1,0) = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}$$

Hence, Taylor's polynomial of degree 2 is

$$P_2(x,y) = 1 - 2(x+1) + y + (x+1)^2 - 2(x+1)y + \frac{y^2}{2}$$

(4) Consider the function:

$$f(x,y,z) = ax^2 + c^2z^2 + \sqrt{2}abxy + ay^2.$$

(a) Find the values of the parameters a, b and c for which the function $f(x,y,z)$ is convex. Find the values of the parameters a, b and c for which the function $f(x,y,z)$ is concave. **1,5 points**

Solution: *The gradient of the function $f(x,y,z)$ is*

$$\nabla f(x,y,z) = \begin{pmatrix} 2ax + \sqrt{2}aby \\ 2ay + \sqrt{2}abx \\ 2c^2z \end{pmatrix}. \tag{1}$$

From (1) we can obtain the Hessian of $f(x,y,z)$:

$$Hf(x,y,z) = \begin{pmatrix} 2a & \sqrt{2}ab & 0 \\ \sqrt{2}ab & 2a & 0 \\ 0 & 0 & 2c^2 \end{pmatrix},$$

which has the following principal minors

$$D_1 = 2a,$$
$$D_2 = 4a^2 - 2a^2b^2,$$
$$D_3 = c^2D_2.$$

The function is convex if the Hessian is positive definite or positive semi-definite, i.e.,

$$D_1 = 2a > 0,$$
$$D_2 = 2a^2(2-b^2) > 0,$$
$$D_3 = c^2D_2 > 0,$$

Note that if $b^2 > 2$, then $D_2 < 0$ and the function can be neither convex nor concave.

(i) *Suppose* $\boxed{c \neq 0}$.

(A) *If $a > 0$ and $b^2 < 2$ then $D_1 > 0$, $D_2 > 0$ and $D_3 > 0$. The Hessian matrix is definite positive and the function is strictly convex.*

(B) *Suppose $a > 0$ and $b^2 = 2$. Exchanging the variables x and z we obtain the function $f(z,y,x) = az^2 + c^2x^2 + \sqrt{2}abzy + ay^2$. The new Hessian matrix is*

$$Hf(x,y,z) = \begin{pmatrix} 2c^2 & 0 & 0 \\ 0 & 2a & \sqrt{2}ab \\ 0 & \sqrt{2}ab & 2a \end{pmatrix},$$

The principal minors are $D_1 = 2c^2 > 0$, $D_2 = 4ac^2 > 0$, $D_3 = 4a^2(2-b^2)c^2 = 0$. We see that Hessian matrix is positive semi-definite. Hence the function is convex.

(C) *If $a = 0$, the function is $f(x,y,z) = cz^2$ and, hence, it is convex.*

(D) *If $a \neq 0$ and $b^2 \neq 2$, then the sign of D_2 is the same as the sign of D_3. The function cannot be concave.*

(ii) *Suppose* $\boxed{c = 0}$. *In this case, the function is $f(x,y,z) = ax^2 + \sqrt{2}abxy + ay^2$ which, considered as a function of three variables cannot be strictly concave or convex, because $f(0,0,z) = 0$.*

(A) *If $a > 0$ and $b^2 < 2$, then $D_1 > 0$, $D_2 > 0$ and $D_3 = 0$. The Hessian matrix is positive semi-definite and the function is convex.*

(B) *If $a < 0$ and $b^2 < 2$, then $D_1 < 0$, $D_2 > 0$ and $D_3 = 0$. The Hessian matrix is negative semidefinite and the function is concave,*

(C) *If $b^2 = 2$, then function is $f(x,y,z) = a(x^2 + y^2 \pm 2xy) = a(x \pm y)^2$ which is concave if $a < 0$ and convex if $a > 0$.*

(b) Consider the case when $a = 1$, $b = 3$. Does the function $f(x,y,z)$ have a local minimum or a local maximum? **0,5 points**

Solution: *If $a = 1$ and $b = 3$, the Hessian is indefinite for all $(x,y,z) \in \mathbb{R}^3$, because $D_1 > 0$, $D_2 < 0$. This, in turn, implies that the only critical point obtained from (1), $x = y = z = 0$, is a saddle point. The function $f(x,y,z)$ does not have a local minimum nor a local maximum.*

(5) Consider the function $f(x, y, z) = x + y + z$ defined on the set

$$A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + 2z^2 = 4, x + y = 2\}$$

(a) Write the Lagrange equations for f on the set A. Compute the points that satisfy those equations and the values of the associated Lagrange multipliers. **1 point**

(b) Using the second order conditions, classify the critical points found in the previous part. Determine the global maxima and minima of the function f on the set A. **1 point**

Solution:

(a) *The objective function f and the restrictions $g_1(x, y, z) = x^2+y^2+2z^2-4$ and $g_2(x, y, z) = x+y-2$ are all of class C^1 (in fact, they are of class C^n for any n). We check now the non-degenerate constraint qualification. We have $\nabla g_1(x, y, z) = (2x, 2y, 4z)$, $\nabla g_1(x, y, z) = (1, 1, 0)$. We compute the rank of the matrix*

$$\text{rank}\begin{pmatrix} 2x & 2y & 4z \\ 1 & 1 & 0 \end{pmatrix} = \text{rank}\begin{pmatrix} 1 & 1 & 0 \\ 2x & 2y & 4z \end{pmatrix} = \text{rank}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2y - 2x & 4z \end{pmatrix}$$

This rank is 2 unless $y = x$ and $z = 0$. But, no the point of the form $(x, x, 0)$ satisfies the constraints $x^2 + y^2 + 2z^2 = 4$, $x + y = 2$. Hence, the assumptions of the Lagrange Theorem are fulfilled. The extreme points of f on the set A are critical points of the Lagrangian

$$L(z, y, z, \lambda) = x + y + z + \lambda(x^2 + y^2 + 2z^2 - 4) + \mu(x + y - 2).$$

The Lagrange equations are:

$$\textit{Lagrange}\begin{cases} \frac{\partial L}{\partial x}(x, y, z) &= 1 + 2\lambda x + \mu = 0 \\ \frac{\partial L}{\partial y}(x, y, z) &= 1 + 2\lambda y + \mu = 0 \\ \frac{\partial L}{\partial z}(x, y, z) &= 1 + 4\lambda z = 0 \\ 4 &= x^2 + y^2 + 2z^2 \\ 2 &= x + y \end{cases}$$

From the third equation we see that $\lambda \neq 0$. Using now the first two equations we see now that $x = y$. Plugging $x = y$ in the last equation we obtain $x = y = 1$. And from the fourth equation we obtain $z = \pm 1$. Hence, we obtain the solutions

$$x = y = z = 1, \quad \lambda = -\frac{1}{4}, \quad \mu = -\frac{1}{2}$$
$$x = y = 1, z = -1, \quad \lambda = \frac{1}{4}, \quad \mu = -\frac{3}{2}$$

(b) *We use the second order conditions to classify critical points. The Hessian matrix of the Lagrangian with respect to (x, y, z) is*

$$\mathcal{H}L_{x,y,z}(x, y, z) = \begin{pmatrix} 2\lambda & 0 & 0 \\ 0 & 2\lambda & 0 \\ 0 & 0 & 4\lambda \end{pmatrix}.$$

At the point $(1, 1, 1)$, the Hessian matrix is negative definite, thus $(1, 1, 1)$ is a local maximum of f on the set A. At the point $(1, 1, -1)$, the Hessian matrix is positive definite, thus $(1, 1, -1)$ is a local minimum of f on the set A.

We can apply Weierstrass' Theorem , since the set A is a compact set and the objective function is continuous. Thus, f has a global maximum and a global minimum on the set A. Since these extreme points satisfy the Lagrange equations, we conclude that $(1, 1, 1)$ corresponds to a global maximum and $(1, 1, -1)$ corresponds to a global minimum of f on the set A.