(1) Consider the following system of linear equations with two parameters $a, b \in \mathbb{R}$

$$\left\{\begin{array}{lcr} x+y+2z & = & 1 \\ 2ax+(3a-1)and+(5a-2)z & = & 2+2a \\ 2ax+(3a-1)y+(5a-2+b^2)z & = & 2a-b+2 \end{array}\right.$$

(a) State the Rouchée–Frobenius Theorem. **0.5 points**

(b) Classify the above system according to the values of a and b. **1 point**

(c) Solve the above system for the values $a = 1$, $b = 1/2$. **0.5 points**

(2) Consider the function

$$f(x,y) = \left\{\begin{array}{rl} \frac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{array}\right.$$

(a) Compute the partial derivatives

$$\frac{\partial f}{\partial x}(0,0) \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0)$$

and the gradient of the function f at the point $(0,0)$. **1 point**

(b) Compute the directional derivative of the function f according to the vector $v = (1,1)$ at the point $p = (0,0)$. Determine if the function f is differentiable at the point $(0,0)$. **1 point**

(3) Consider the function $f(x,y) = y^3 - x^3 + 3x^2 + 3y^2$.

(a) Compute and classify the critical points (if any) of the function f in the set $\mathbb{R}^2$. **1 point**

(b) Find the largest open subset $S \subset \mathbb{R}^2$ where the function f is convex. Compute and classify the critical points (if any) of the function f in the set S. **1 point**

(4) Consider the function $f(x,y) = x^2 \ln y$.

(a) Compute the plane tangent to the graph of the function f at the point $p = (1,1,0)$. **1 point**

(b) Compute the Taylor polynomial of order 2 of the function f at the point $p = (1,1)$. **1 point**

(5) Let $f(x,y,z) = x + z$ and consider the sphere with equation $x^2 + y^2 + z^2 = 1$.

(a) Verify that the assumptions of Lagrange's Theorem hold. Write the Lagrange equations and obtain the solutions of those equations. **1 point**

(b) Determine the extreme points of f on the sphere. Determine if those points are local or global extreme points. Justify the answer. **1 point**