(1) Consider the following system of linear equations with two parameters $a, b \in \mathbb{R}$

$$\left\{\begin{array}{lcc} x+y+2z & = & 1 \\ 2ax+(3a-1)y+(5a-2)z & = & 2+2a \\ 2ax+(3a-1)y+(5a-2+b^2)z & = & 2a-b+2 \end{array}\right.$$

(a) State the Rouchée–Frobenius Theorem. **0.5 points**

(b) Classify the above system according to the values of a and b. **1 point**

Solution: *The augmented matrix associated to the system is*

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2a & 3a-1 & 5a-2 & 2a+2 \\ 2a & 3a-1 & b^2+5a-2 & 2a-b+2 \end{pmatrix}$$

After elementary row operations, we obtain

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2a & 3a-1 & 5a-2 & 2a+2 \\ 2a & 3a-1 & b^2+5a-2 & 2a-b+2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & a-1 & a-2 & 2 \\ 0 & a-1 & b^2+a-2 & 2-b \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & a-1 & a-2 & 2 \\ 0 & 0 & b^2 & -b \end{pmatrix}$$

and the original system of equations is equivalent to another one whose augmented matrix is

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & a-1 & a-2 & 2 \\ 0 & 0 & b^2 & -b \end{pmatrix}$$

We compute the determinant expanding the last row of the matrix A and we obtain $\det A = b^2(a-1)$. *We conclude that,if $a \neq 1$ and $b \neq 0$, the system has a unique solution.*
Suppose now that $b = 0$. Then, the original system is equivalent to another system of linear equations whose associated matrix is

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & a-1 & a-2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And we see that, when $b = 0$, $\text{rank}(A) = \text{rank}(A|b) = 2$ *and the system is undetermined with $3-2=1$ parameter, for every value of the parameter a. Suppose now that $a = 1$. The system is equivalent to a system of linear equations whose augmented matrix is*

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & b^2 & -b \end{pmatrix}$$

And we see that $\text{rank}(A) = 2$. *Adding the second row multiplied by b^2 to the third row we obtain the matrix*

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & b(2b-1) \end{pmatrix}$$

We conclude that if $a = 1$ and $b \in \mathbb{R} \setminus \{0, 1/2\}$ the system is inconsistent. Whereas if $a = 1$ and $b = 0$ or $b = 1/2$, the system is consistent and has infinitely many solutions which can be described by $3-2=1$ parameter.

(c) Solve the above system for the values $a = 1$, $b = 1/2$. **0.5 points**

Solution: *The system is equivalent to the following system of linear equations*

$$\left\{\begin{array}{lcc} x+and+2z & = & 1 \\ -z & = & 2 \end{array}\right.$$

choosing y as the parameter, the set of solutions is $\{(5-y, y, -2) : y \in \mathbb{R}\}$.

(2) Consider the function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Compute the partial derivatives

$$\frac{\partial f}{\partial x}(0,0) \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0)$$

and the gradient of the function f at the point $(0,0)$. **1 point**

Solution: *Since, for $x, y \neq 0$, $f(x,0) = f(0,y) = f(0,0) = 0$ we have that*

$$\lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0$$

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$$

and $\nabla f(0,0) = (0,0)$.

(b) Compute the directional derivative of the function f according to the vector $v = (1,1)$ at the point $p = (0,0)$. Determine if the function f is differentiable at the point $(0,0)$. **1 point**

Solution: *For $t \neq 0$ we have that*

$$f(p+tv) = f((0,0) + t(1,1)) = f(t,t) = \frac{t^3}{2t^2} = \frac{t}{2}$$

Therefore,

$$D_v(p) = \lim_{t\to 0} \frac{f(p+tv) - f(p)}{t} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$$

Since,

$$D_v(p) \neq \nabla f(0,0) \cdot v$$

the function f is not differentiable at the point $(0,0)$.

(3) Consider the function $f(x, y) = y^3 - x^3 + 3x^2 + 3y^2$.

(a) Compute and classify the critical points (if any) of the function f in the set $\mathbb{R}^2$. **1 point**

Solution: *The gradient of f is*

$$(6x - 3x^2, 3y^2 + 6y)$$

The equations that define the critical points are

$$\begin{aligned} 0 &= 6x - 3x^2 \\ 0 &= 3y^2 + 6y \end{aligned}$$

The solutions are $(0, 0)$, $(0, -2)$, $(2, -2)$ and $(2, 0)$. the Hessian matrix of the function f is

$$H(x, y) = \begin{pmatrix} 6 - 6x & 0 \\ 0 & 6y + 6 \end{pmatrix}$$

We see that

$$H(0,0) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \quad H(0,-2) = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}, \quad H(2,-2) = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}, \quad H(2,0) = \begin{pmatrix} -6 & 0 \\ 0 & 6 \end{pmatrix}$$

Using the second order conditions, we conclude that at the point $(0, 0)$ the function attains a local minimum, at $(2, -2)$ it attains a local maximum and $(2, 0)$ and $(0, -2)$ are saddle points Finally, $f(0, y) = y^3 + 3y^2$ and we see that $\lim_{y\to\infty} f(0, y) = +\infty$, $\lim_{y\to-\infty} f(0, y) = -\infty$. So, there are no global maxima or minima in the set $\mathbb{R}^2$.

(b) Find the largest open subset $S \subset \mathbb{R}^2$ where the function f is convex. Compute and classify the critical points (if any) of the function f in the set S. **1 point**

Solution: *The Hessian matrix of the function f is*

$$Hf(x, y) = \begin{pmatrix} 6 - 6x & 0 \\ 0 & 6y + 6 \end{pmatrix}$$

We see that

$$D_1 = 6(1 - x), \quad D_2 = 36(1 - x)(1 + y)$$

Thus, $D_1 > 0$ if and only if $x < 1$. Assuming that $x < 1$, we see that $D_2 > 0$ if and only if $y > -1$. Therefore,

$$S = \{(x, y) \in \mathbb{R}^2 : x < 1, y > -1\}$$

We see that the only critical point in S is $(0, 0) \in S$. By the local-global Theorem, $(0, 0)$ corresponds to global minimum of f on S.

(4) Consider the function $f(x,y) = x^2 \ln y$.

(a) Compute the plane tangent to the graph of the function f at the point $p = (1,1,0)$. **1 point**

Solution: *Since, $f(1,1) = 0$, the point p is in the graph of the function f. On the other hand, $\nabla f(1,1) = (0,1)$. The equation of the tangent plane is $z = f(1,1) + \nabla f(1,1) \cdot (x-1, y-1) = 0 + (0,1) \cdot (x-1, y-1)$, that is,*

$$z = y - 1$$

(b) Compute the Taylor polynomial of order 2 of the function f at the point $p = (1,1)$. **1 point**

Solution: *The Hessian matrix of f is*

$$Hf(x,y) = \begin{pmatrix} 2\ln y & \frac{2x}{y} \\ \frac{2x}{y} & -\frac{x^2}{y^2} \end{pmatrix}$$

and

$$Hf(1,1) = \begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}$$

The Taylor polynomial of order 2 is

$$\begin{aligned} P_2(x,y) &= f(1,1) + \nabla f(1,1) \cdot (x-1, y-1) + \frac{1}{2} \begin{pmatrix} x-1 & y-1 \end{pmatrix} Hf(1,1) \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \\ &= y - 1 + 2(x-1)(y-1) - \frac{1}{2}(y-1)^2 \end{aligned}$$

(5) Let $f(x,y,z) = x + z$ and consider the sphere with equation $x^2 + y^2 + z^2 = 1$.

(a) Verify that the assumptions of Lagrange's Theorem hold. Write the Lagrange equations and obtain the solutions of those equations. **1 point**

Solution: *The objetive function f and the restriction $h(x,y,z) = x^2 + y^2 + z^2 - 1$ are both of class $\mathcal{C}^1$ in $\mathbb{R}^3$ (if fact, they are of class $\mathcal{C}^n$, for any n). In addition, the gradient of h is $\nabla h(x,y,z) = (2x, 2y, 2z)$, does not vanish at any point of the sphere. Hence, the assumptions of the Lagrange multiplier Theorem are fulfilled. And the extreme points of f on the sphere are also critical points of Lagrange's function:*

$$L_\lambda(x,y,z) = x + z + \lambda(x^2 + y^2 + z^2 - 1),$$

for some $\lambda \in \mathbb{R}$. Hence, the possible extreme points of f in the sphere satisfy the following equations:

$$\begin{cases} \frac{\partial L_\lambda}{\partial x}(x,y,z) & = 1 + 2x\lambda = 0 \\ \frac{\partial L_\lambda}{\partial y}(x,y,z) & = 2y\lambda = 0 \\ \frac{\partial L_\lambda}{\partial z}(x,y,z) & = 1 + 2z\lambda = 0 \\ x^2 + y^2 + z^2 & = 1 \end{cases}$$

This is a system of four non-linear equations with four unknowns. First, note that $\lambda \neq 0$. Otherwise the first and third equation would yield a contradiction. Now, from the second equation we obtain $y = 0$. On the other hand, from the first and third equation we get that $x = z$. And plugging these last two equalities in the last equation we conclude that $x = z = \pm\frac{1}{\sqrt{2}}$. Hence, the critical points are $P_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, for $\lambda = -1/\sqrt{2}$ and $P_2 = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$ for $\lambda = 1/\sqrt{2}$.

(b) Determine the extreme points of f on the sphere. Determine if those points are local or global extreme points. Justify the answer. **1 point**

Solution: *We use the second order conditions to classify the critical points. The Hessian matrix of the Lagrangian is*

$$\mathcal{H}L_\lambda(x,y,z) = \begin{pmatrix} 2\lambda & 0 & 0 \\ 0 & 2\lambda & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}$$

At the point P_1 the Hessian matrix is negative definite. Hence, the function f attains a local maximum at the point P_1. At the point P_2 the Hessian matrix is positive definite. Hence, the function f attains a local minimum at the point P_2.

On the other hand, we note that the objective function is continuous on the sphere, which is a compact set. Therefore Weierstras' Theorem garanties that f attains a maximum and a minimum value on the sphere. Since these extreme points satisfy the Lagrange equations, we conclude that P_1 corresponds to a global maximum and P_2 corresponds to a global minimum of f on the sphere.