UNIVERSIDAD CARLOS III de MADRID

Department of Economics

Final Exam of Mathematics II. June 2016.

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(1) Given the following system of linear equations with a parameter $a \in \mathbb{R}$

$$\begin{cases} ax + y + z &= 1 \\ x + ay + z &= a \\ x + y + az &= a^2 \end{cases}$$

(a) Classify the system according to the values of a. **1 point**

Solution: Using Gauss' method, the row elementary operations, we have obtained the following augmented matrix of a linear system of equations equivalent to the given system:

$$\begin{pmatrix} 1 & 1 & a & a^2 \\ 0 & a-1 & 1-a & a-a^2 \\ 0 & 0 & -a^2-a+2 & -a^3-a^2+a+1 \end{pmatrix}$$

Firstly, if $a \neq 1$ and $a \neq -2$ the linear system is consistent and determined, with only one solution. Secondly, suppose that $a = 1$, the linear system is equivalent to another one whose augmented matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

Then the system is consistent and underdetermined and the solution needs 2 different parameters. Thirdly, if $a = -2$ the linear system is equivalent to another one whose augmented matrix is

$$\begin{pmatrix} 1 & 1 & -2 & 4 \\ 0 & -3 & 3 & -6 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

and the linear system is inconsistent, it has no solution.

(b) Solve the above system for the value of the parameter $a = 1$. **1 point**

Solution: As we said above, in this case the linear system is equivalent to

$$x + y + z = 1$$

If we choose x, y as the real parameters the solution will be $\{(x, y, 1-x-y) : x, y \in \mathbb{R}\}$.

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(2)

(a) Show that the following system of equations

$$\begin{aligned} y^2 + z^2 - x^2 + 2 &= 0 \\ yz + xz - xy - 1 &= 0 \end{aligned}$$

defines two functions $y = y(x)$, $z = z(x)$ in a neighborhood of the point $(x, y, z) = (2, 1, 1)$. Compute $y'(2)$, $z'(2)$. **1 point**

Solution: Defining the functions $f_1(x, y, z) = y^2 + z^2 - x^2 + 2$, $f_2(x, y, z) = yz + xz - xy - 1$. We calculate

$$\frac{\partial(f_1, f_2)}{\partial y \partial z} = \begin{vmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{vmatrix} = \begin{vmatrix} 2y & 2z \\ z-x & x+y \end{vmatrix} = 2(x+y-z)(y+z)$$

and we obtain $J(f_1, f_2)(2, 1, 1) = 8$, using the implicit function theorem we know that there are two implicit functions $y = y(x)$, $z = z(x)$ defined by the system of equations on a neighborhood of the point $(x, y, z) = (2, 1, 1)$. Now if we differentiate implicitly the system with respect to x we obtain:

$$\begin{aligned} 2y(x)y'(x) + 2z(x)z'(x) - 2x &= 0 \\ y'x)z(x) + y(x)z'(x) + z(x) + xz'(x) - y(x) - xy'(x) &= 0 \end{aligned}$$

Evaluating these equations at the values $x = 2$, $y(2) = 1$, $z(2) = 1$ we get

$$\begin{aligned} 2y'(2) + 2z'(2) - 4 &= 0 \\ 3z'(2) - y'(2) &= 0 \end{aligned}$$

and solving the linear system, our solutions are $y'(2) = \frac{3}{2}$, $z'(2) = \frac{1}{2}$.

(b) Consider the functions $F(x, y, z) = xz - y^2$ and $G(x) = F(x, y(x), z(x))$. Compute $G'(2)$. **1 point**

Solution: Differentiating the new function implicitly we obtain $G'(x) = -2y(x)y'(x) + xz'(x) + z(x)$. Therefore, $G(2) = -2y(2)y'(2) + 2z'(2) + z(2) = -1$.

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(3) Consider the function $f(x,y) = \frac{x^2}{y} + ay^2$ with $a \in \mathbb{R}$ defined on the open set $D = \{(x,y) \in \mathbb{R}^2 : y > 0\}$.

(a) Study the convexity of the function f in the set A, depending on the values of the parameter a. **1 point**

Solution: We need the hessian matrix in order to calculate the sign of quadratic form related to it. Firstly we calculate the gradient vector of the function:

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} \frac{2x}{y} & \frac{-x^2}{y^2} + 2ay \end{pmatrix}$$

and secondly, calculating the second order derivatives we get the hessian

$$Hf(x,y) \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial y \partial x}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} \frac{2}{y} & \frac{-2x}{y^2} \\ \frac{-2x}{y^2} & \frac{2x^2}{y^3} + 2a \end{pmatrix}$$

Using the leading minors method to calculate its sign: $D_1 = \frac{2}{y}$ and $D_2 = \frac{2}{y}\left(\frac{2x^2}{y^3} + 2a\right) - \left(\frac{-2x}{y^2}\right)^2 = \frac{4a}{y}$ and we get: $\forall (x,y) \in A$ and $\forall a \geq 0$ $D_1 > 0$ and $D_2 \geq 0$ then Hf is positive definite or positive semidefinite, that means f is convex. $\forall (x,y) \in A$ and $\forall a < 0$ $D_1 > 0$ and $D_2 < 0$ then, Hf is indefinite and f is neither convex nor concave.

(b) Compute the Taylor polynomial of degree 2 of the function f at the point $p = (0,1)$. **1 point**

Solution: The second order Taylor's Polynomial is

$$\begin{aligned} p_2(x,y) &= f(0,1) + \nabla f(0,1) \cdot (x, y-1) + \frac{1}{2} \begin{pmatrix} x & y-1 \end{pmatrix} Hf(0,1) \begin{pmatrix} x \\ y-1 \end{pmatrix} \\ &= a + 2a(y-1) + x^2 \; + a(y-1)^2 \end{aligned}$$

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(4) Consider the function $f(x,y,z) = x - y + z$ and the set $A = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 9, x + z = 4\}$.

(a) Compute the Lagrange equations that determine the extreme points of the function f in the set A. Compute the points that satisfy the Lagrange equations and the values of the corresponding Lagrange multipliers at each of the points. **1 point**

Solution: The Lagragian function of the problem is $L = x - y + z - \lambda\left(x^2 + y^2 + z^2 - 9\right) - \mu(x + z - 4)$. So, Lagrange equations are:

$$1 - 2\lambda x - \mu = 0, \quad -1 - 2\lambda y = 0, \quad 1 - 2\lambda z - \mu = 0, \quad x + z = 4 \quad x^2 + y^2 + z^2 = 9$$

Whose solutions are

$$x = z = 2, y = -1, \lambda = \frac{1}{2}, \mu = -1$$

and

$$x = z = 2, y = 1, \lambda = -\frac{1}{2}, \mu = 3$$

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(b) Using the second order conditions, classify the solutions found in the previous part into maxima, minima and local points. Can you say if any of the local maxima and/or maxima is a global extreme point on the set A? Justify adequately your answers. **1 point**

Solution: The Hessian matrix is

$$H(x,y,z;\lambda,\mu) = \begin{pmatrix} -2\lambda & 0 & 0 \\ 0 & -2\lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}$$

On the one hand, at the point $x = z = 2, y = -1, \lambda = \frac{1}{2}, \mu = -1$, we obtain

$$H(2,-1,2;1/2,-1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and its sign is negative definite. Thus, the point $x = z = 2, y = -1$ is a local minimum. On the other hand, at the point $x = z = 2, y = 1, \lambda = -\frac{1}{2}, \mu = 3$, we get

$$H(2,1,2,;1/2,3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that is positive definite. So, the point $x = z = 2, y = 1$ is a local minimum. Since the set of points A is regular, compact and the function f is continuous, then applying Weierstrass' theorem we can state that the calculated critical points are indeed the global extreme points of f on A.

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(5) Consider the function $f(x, y) = x^4 + y^3 - 2a^2x^2 - 3y$ with $a \in \mathbb{R}$, $a \neq 0$.

(a) Determine the critical points of the function f in the set $\mathbb{R}^2$. **1 point**

Solution: The gradient vector of f is

$$\left(-4x(a-x)(a+x), 3\left(y^2-1\right)\right)$$

and the critical points are defined by these equations:

$$\begin{aligned} 0 &= x(a-x)(a+x) \\ 0 &= y^2 - 1 \end{aligned}$$

The solutions of this system are: $(0, \pm 1)$, $(\pm a, \pm 1)$.

(b) Classify the critical points of the previous part into (local and/or global) maximum and/or minimum points and saddle points. **1 point**

Solution: The Hessian matrix is

$$H(x, y) = \begin{pmatrix} 12x^2 - 4a^2 & 0 \\ 0 & 6y \end{pmatrix}$$

And we obtain for all of those points,

$$H(0, -1) = \begin{pmatrix} -4a^2 & 0 \\ 0 & -6 \end{pmatrix}, \quad H(-a, -1) = \begin{pmatrix} 8a^2 & 0 \\ 0 & -6 \end{pmatrix}, \quad H(a, -1) = \begin{pmatrix} 8a^2 & 0 \\ 0 & -6 \end{pmatrix}$$

$$H(0, 1) = \begin{pmatrix} -4a^2 & 0 \\ 0 & 6 \end{pmatrix}, \quad H(-a, 1) = \begin{pmatrix} 8a^2 & 0 \\ 0 & 6 \end{pmatrix}, \quad H(a, 1) = \begin{pmatrix} 8a^2 & 0 \\ 0 & 6 \end{pmatrix}$$

Therefore, the point $(0, -1)$ is a local minimum, all the points $(\pm a, 1)$ are local minima and the points $(-a, -1)$, $(a, -1)$, $(0, 1)$ are saddle points. Furthermore, we can see that $f(x, 0) = x^4 - 2a^2x^2$. Then $\lim_{x \to \infty} f(x, 0) = +\infty$ and there is no global maximum. Also, we observe that $f(0, y) = y^3 - 3y$. Then $\lim_{x \to -\infty} f(0, y) = -\infty$ and there is no global minimum.