1

Consider the following linear system that depends on parameters $a, b \in \mathbb{R}$:

$$\left\{\begin{array}{cccc} x & +2y & +2z & = & -4 \\ 2x & +6y & +3z & = & -8 \\ x & +6y & +az & = & -b-4 \\ -x & -2y & +(a-2)z & = & -a+4 \end{array}\right.$$

(a) (6 points) Discuss the system for the all possible values of the parameters a, b.

(b) (4 points) Calculate the solution on the system for values of the parameters a and b for which the system admits a unique solution.

Solution:

(a) Transform the system's augmented matrix A^* to an equivalent echelon matrix by using Gaussian elimination.

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -4 \\ 2 & 6 & 3 & -8 \\ 1 & 6 & a & -b-4 \\ -1 & -2 & a-2 & -a+4 \end{array}\right) \underset{\sim}{\overset{r_2-2r_1}{\underset{r_4+r_1}{r_3-r_1}}} \left(\begin{array}{ccc|c} 1 & 2 & 2 & -4 \\ 0 & 2 & -1 & 0 \\ 0 & 4 & a-2 & -b \\ 0 & 0 & a & -a \end{array}\right) \underset{\sim}{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 2 & -4 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & a & -a \end{array}\right)$$

$$\underset{\sim}{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 2 & -4 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & 0 & b-a \end{array}\right).$$

When $a \neq b$ the system has no solution; when $a = b$ and $a \neq 0$, the system admits a unique solution; when $a = b = 0$, the system has infinitely many solutions.

(b) The system has a unique solution when $a = b$ and $a \neq 0$. From the third equation of the echelon system we find $z = -1$, then from the second one $2y - z = 0$, or $y = -\frac{1}{2}$ and form the first equation $x + 2y + 2z = -4$, or $x = -4 + 1 + 2 = -1$. The solution is thus $x = -1$, $y = -\frac{1}{2}$, $z = -1$.

2

Let consider the set $A = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1, x \neq y\}$ and the function $f(x, y) = x^2 - 4y$.

(a) (3 points) Draw the set A. Discuss whether the assumptions of the Theorem of Weierstrass are fulfilled. Justify your answer.

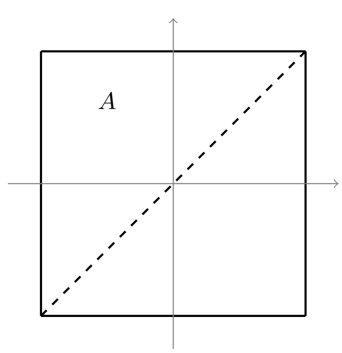
(b) (7 points) Draw the level curves of f. Using this information, find the points where the global maximum and global minimum values of f in A are attained, if they exist.

Solution:

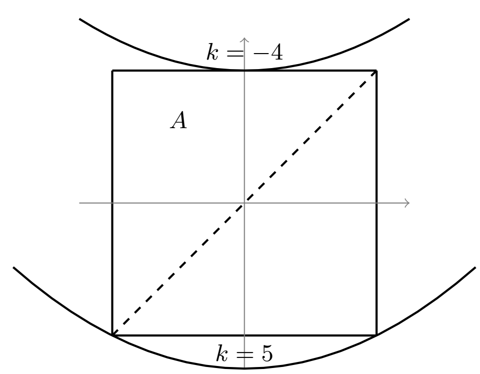
(a) See below the representation of A. The set A is bounded but not closed, as it does not contain all its boundary points (more precisely, the diagonal points (x, x)). In consequence, one of the hypothesis of Weierstrass Theorem fails.

(b) The k-level curve of f, $y = \frac{x^2}{4} - \frac{k}{4}$, is a convex parabola with vertex on the vertical axis, with coordinates $(0, -k/4)$; k increases as the vertex moves down, so we infer that the lowest parabola touching A provides the global maximum of f in A. In the same way, the highest parabola touching A provides the global minimum of f in A. See the figure below, right. In the first case the parabola meets the square $[-1, 1] \times [-1, 1]$ at $(1, -1)$ and $(-1, -1)$. Only $(1, -1)$ belongs to A. Note that k is $1^2 - 4(-1) = 5 = k$. Hence the global maximum of f in A is $(1, -1)$ and the maximum value of f is 5. In the second case, the parabola is tangent to $y = 1$, that is, it touches A at a unique point, which is $(0, 1)$. Note that k is $0^2 - 4 = k = -4$. Hence the global minimum of f in A is $(0, 1)$ and the minimum value is -4.

Set



Level curves



3

Let consider the function $f(x,y) = -\dfrac{x^2}{2} + 4xy - y^4$.

(a) (5 points) Determine the largest open and convex sets of $\mathbb{R}^2$ where the function f is convex or concave.

(b) (5 points) Calculate the local extremum points of f if they exist.

Solution:

(a) The function f is of class $\mathcal{C}^2$ in $\mathbb{R}^2$, since it is a polynomial. The gradient of f is

$$\nabla f(x,y) = (-x + 4y, 4x - 4y^3).$$

The Hessiana matrix of f is

$$\begin{pmatrix} -1 & 4 \\ 4 & -12y^2 \end{pmatrix}.$$

Principal minors are $D_1 = -1 < 0$ and $D_2 = 12y^2 - 16$. Hence f is not convex in any region of $\mathbb{R}^2$. The determinant is nonnegative iff $y^2 \geq \dfrac{4}{3}$, that is, when $|y| \geq \dfrac{2}{\sqrt{3}}$. In consequence, f is concave (strictly) in each one of the following convex and open sets: $C_1 = \left\{(x,y) \in \mathbb{R}^2 \,:\, y > \dfrac{2}{\sqrt{3}}\right\}$ and $C_2 = \left\{(x,y) \in \mathbb{R}^2 \,:\, y < -\dfrac{2}{\sqrt{3}}\right\}$. Note that $C_1 \cup C_2$ is open but not convex!

(b) The critical points of f solve the system

$$\left.\begin{array}{rcl} -x + 4y & = & 0 \\ 4x - 4y^3 & = & 0 \end{array}\right\}.$$

Plugging $x = 4y$ into the second equation we get $16y - 4y^3 = 4y(4 - y^2) = 0$, with solutions 0 and ± 2, thereby the critical points are $(0,0)$, $(8,2)$ and $(-8,-2)$. It is immediate that $(0,0)$ is a saddle and both $(8,2)$ and $(-8,-2)$ are local maximum of f.

4

Let consider the problem of optimizing the function $f(x,y) = x$ on the set $A = \{(x,y) : x^2 + xy - y^2 = 20\}$.

(a) (4 points) Find the Lagrange equations and find the critical points, as well as the corresponding Lagrange multiplier.

(b) (6 points) Classify the critical points obtained in part (a) above.

Solution:

(a) The Lagrangian is

$$L(x,y,\lambda) = x + \lambda(20 - x^2 - xy + y^2).$$

and the Lagrange equations are

$$\begin{cases} L'_x &= 1 + \lambda(-2x - y) = 0 \\ L'_y &= \lambda(-x + 2y) = 0 \\ L'_\lambda &= 20 - x^2 - xy + y^2 0, \end{cases}$$

from which we conclude that $\lambda \neq 0$ and hence $x = 2y$. Plugging this into the constraint we get $4y^2 + 2y^2 - y^2 = 20$, or $y = \pm 2$. The critical points are thus $(4,2)$ and $(-4,-2)$. The multiplier λ is found by substituting each of the above points into the equation $L'_x = 0$, obtaining $\lambda = 1/10$ for $(4,2)$ and $\lambda = -1/10$ for $(-4,-2)$.

(b) The Hessian matrix of the Lagrangian with respect to the variables (x,y) is

$$\mathcal{H}_{(x,y)}L(x,y,\lambda) = \begin{pmatrix} -2\lambda & -\lambda \\ -\lambda & 2\lambda \end{pmatrix}.$$

Substituting the first critical point we obtain

$$\mathcal{H}_{(x,y)}L(4,2,1/10) = \begin{pmatrix} -1/5 & 1/10 \\ 1/10 & 1/5 \end{pmatrix}$$

which is indefinite. Hence we proceed to study the sign of the quadratic form restricted to the tangent subspace to the constraint $g(x,y) = x^2 + xy - y^2 - 20$ at the point $P = (x_0, y_0)$

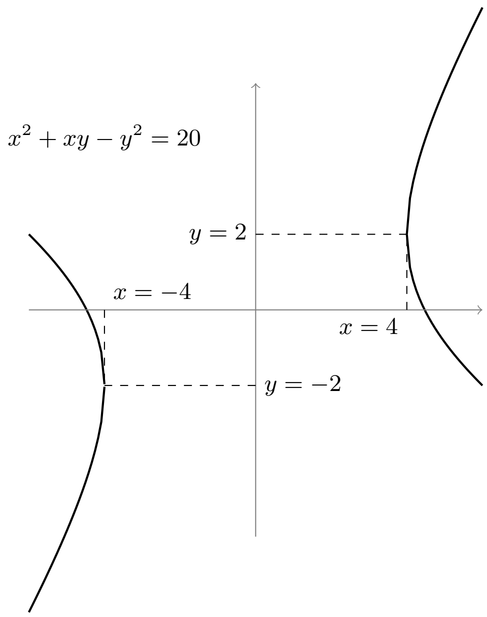
$$T_P = \{(h,k) \in \mathbb{R}^2 : \nabla g(P) \cdot (h,k) = 0\},$$

where P is a critical point. At $P = (4,2)$, $T_{(4,2)} = \{(h,k) : (10,0) \cdot (h,k) = 0\} = \{h = 0\}$, hence the restriction of the quadratic form is simply $\frac{1}{5}k^2$, which is positive definite, so $(4,2)$ is local minimum of f in A. Analogously,

$$\mathcal{H}_{(x,y)}L(-4,-2,-1/10) = \begin{pmatrix} 1/5 & 1/10 \\ 1/10 & -1/5 \end{pmatrix}$$

is also indefinite. The tangent space at $(-4,-2)$ is the same as before, $T_{(-4,-2)} = \{h = 0\}$, and the restriction gives $-\frac{1}{5}k^2$, negative definite, so $(-4,-2)$ is a local maximum of f in A.

Note: As we know, there is no contradiction in that the local minimum provides a higher value to f than the local maximum. The constraint is represented in the figure below.


$x^2 + xy - y^2 = 20$
$y = 2$
$x = -4$
$x = 4$
$y = -2$

5

Let consider the equation

$$zx^2 + e^{-xz} + yz = 0.$$

(a) (4 points) Determine whether the Implicit Function Theorem can be applied to the equation to get a function $z(x, y)$ of class $\mathcal{C}^1$ around the point $(x, y) = (0, -1)$ for which $z(0, -1) = 1$.

(b) (6 points) Calculate the de Taylor polynomial of order one of the function $z(x, y)$ at the point $(x, y) = (0, -1)$.

Solution:

(a) Let us check the assumptions. To this end, let us denote $f(x, y, z) := zx^2 + e^{xz} + yz$.

- $f \in \mathcal{C}^1$ since it is sum of elementary functions.
- $f(0, -1, 1) = 0 + 1 - 1 = 0$.
- Moreover,

$$\frac{\partial f}{\partial z}(0, -1, 1) = x^2 - xe^{xz} + y|_{(0,-1,1)} = -1 \neq 0.$$

Hence the equation defines z as a function of (x, y) of class $\mathcal{C}^1$ around the point $(0, -1, 1)$ with $z(0, -1) = 1$.

(b) Take the derivative with respect to x and with respect to y to get two equations ($\frac{\partial z}{\partial x}(x, y)$ and $\frac{\partial z}{\partial y}(x, y)$ are replaced by z'_x and z'_y respectively to simplify the writing).

$$\left.\begin{array}{rcl} z'_x x^2 + 2xz - (z + xz'_x)e^{-xz} + yz'_x & = & 0 \\ z'_y x^2 - xz'_y e^{-xz} + z + yz'_y & = & 0 \end{array}\right\}$$

Plugging $(x, y, z) = (0, -1, 1)$ into the equations we get the system

$$\left.\begin{array}{rcl} -1 - z'_x(0, -1) & = & 0 \\ 1 - z'_y(0, -1) & = & 0 \end{array}\right\}$$

which solution is $z'_x(0, -1) = -1$ and $z'_y(0, -1) = 1$. The Taylor polynomial of first order of $z(x, y)$ in $(0, -1)$ is

$$z(0, -1) + z'_x(0, -1)x + z'_y(0, -1)(y + 1) = 1 - x + y + 1 = 2 - x + y.$$

6

Let consider the function

$$f(x,y) = \begin{cases} \dfrac{y^2}{x} & \text{si } x \neq 0, y \in \mathbb{R}, \\ 0 & \text{si } x = 0, y \in \mathbb{R}. \end{cases}$$

(a) (4 points) Study the continuity of f at $(0,0)$.

(b) (6 points) Given $a, b \in \mathbb{R}$ with $a > 0$ and $a^2 + b^2 = 1$, calculate the directional derivative of f in $(0,0)$ along the direction $\mathbf{v} = (a,b)$, that is, compute $D_{(a,b)}f(0,0)$. When does this directional derivative take the minimum possible value?

Solution:

(a) The function is not continuous in $(0,0)$. For instance, by taking limits in $(0,0)$ along the curve $y = m\sqrt{x}$, $x > 0$, we have

$$\lim_{x \to 0^+, y = m\sqrt{x}} f(x,y) = \lim_{x \to 0^+} \frac{m^2 x}{x} = m^2$$

that depends on m, thus the limit of f in $(0,0)$ does not exists and then f is not continuous in $(0,0)$.

(b)

$$D_{(a,b)}f(0,0) = \lim_{t \to 0} \frac{f\big((0,0) + t(a,b)\big) - f(0,0)}{t} = \lim_{t \to 0} \frac{f(ta,tb) - f(0,0)}{t} = \lim_{t \to 0} \frac{t^2 b^2}{t^2 a} = \frac{b^2}{a} = \frac{1 - a^2}{a}.$$

Since $0 < a \leq 1$, the directional derivative is minimized at $a = 1$, that is, along the direction $(a,b) = (1,0)$.