UC3M
Mathematics for Economics II
Final Exam, 06/27/2014 Solutions

1

Consider the matrix

$$\begin{pmatrix} 0 & 3 & 4 \\ 1 & -4 & -5 \\ -1 & 3 & 4 \end{pmatrix}.$$

(a) (5 points) Prove that $A^3 + I = O$, where I is the identity matrix and O the null matrix.

(b) (5 points) By using part (a), calculate A^{10}.

Solution:

(a) It is easy to see that $A^3 = -I$, hence $A^3 + I = O$.

(b) From part (a) we have $A^{10} = (A^3)^3 A = (-I)^3 A = -A$.

$\boxed{2}$

Let the set $A = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and for $a \in \mathbb{R}$ consider the function $f(x, y) = \ln(x + y - a)$.

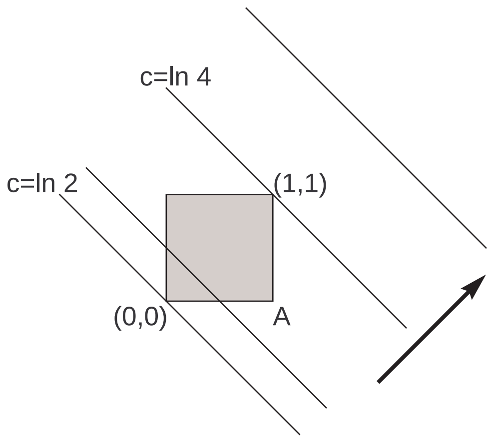
(a) (4 points) Study for which values of a the hypotheses of the Theorem of Weierstrass hold for the function f and the set A.

(b) (6 points) Consider the cases (i) $a = -2$ and (ii) $a = 0$. Draw the set A and the level curves of f in both cases. Using this information, find the points where the global maximum and global minimum value of f are obtained in A, and the value of f in those points.

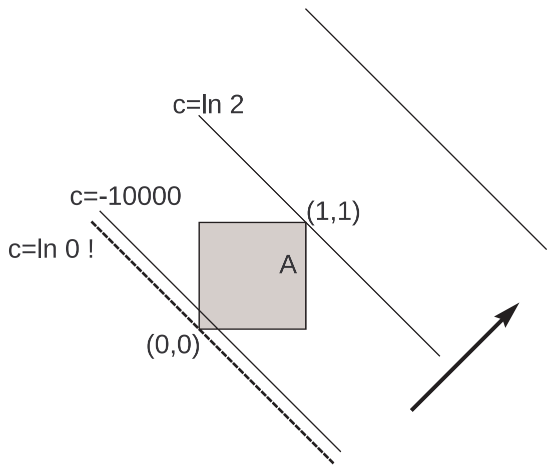
Solution:

(a) A is closed and bounded (hence compact), thus we only have to check that f is continuous on A. The logarithm is continuous in its domain, which are the positive numbers. Hence we have to impose $x + y - a > 0$ for any $(x, y) \in A$. Given that $A = \{0 \leq x \leq 1, 0 \leq y \leq 1\}$, we need $a < 0$ Thus the condition for the theorem to be applicable is $a < 0$.

(b) $\nabla f(x, y) = (\frac{1}{x+y-a}, \frac{1}{x+y-a})$, hence the direction of increase of f at any point of its domain is $(1, 1)$. In the case (i) $a = -2$, f is continuous on A, hence global extrema exist. The global maximum of f on A is attained at the north east corner of set A, that is, at point $(1, 1)$ and the value is $\ln 4$. The global minimum is attained at $(0, 0)$ and the value is $\ln 2$. In the case (ii) $a = 0$, f is not defined at $(0, 0) \in A$, hence we cannot apply the Theorem of Weierstrass. Actually, f is not bounded below on A, as $\lim_{(x,y)\to(0,0)} \ln(x + y) = -\infty$ and there is no minimum. The global maximum is obtained the same as above and the value is $\ln 2$.



Level curves for $a = -2$



Level curves for $a = 0$, notice the dotted line is not a level curve.

$\boxed{3}$

Let $a \in \mathbb{R}$ and consider the function $f(x, y) = ax^2 + ay^2 + xy + 3x - 3y$.

(a) (4 points) Calculate the value or values of a so that f is strictly concave or strictly convex.

(b) (6 points) Let $a = 2$. Calculate the critical points of f and classify them into local or global maxima, minima and saddle points.

Solution:

(a) The gradient of f is $\nabla f(x, y) = (2ax + y + 3, 2ay + x - 3)$ and the Hessian matrix is

$$Hf(x, y) = \begin{pmatrix} 2a & 1 \\ 1 & 2a \end{pmatrix},$$

which is independent of (x, y). The principal minors are $D_1 = 2a$ and $D_2 = 4a^2 - 1$. We have:

- $D_1 > 0$, $D_2 > 0$ iff $a > 0$ and $|a| > \frac{1}{2}$: Positive definite.
- $D_1 < 0$, $D_2 > 0$ iff $a < 0$ and $|a| > \frac{1}{2}$: Negative definite.

Hence, when $a > \frac{1}{2}$, f is strictly convex in $\mathbb{R}^2$ and when $a < -\frac{1}{2}$, f is strictly concave in $\mathbb{R}^2$.

(b) Let $a = 2$. The critical points of f are the solutions of $\nabla f(x, y) = (4x + y + 3, 4y + x - 3) = (0, 0)$. There is only one solution, $(-1, 1)$. Since that $a > \frac{1}{2}$, f is strictly convex by part (a), hence $(-1, 1)$ is a strict global minimum of f.

$\boxed{4}$

A student has to take an exam on two subjects, A and B. She disposes of 25 hours to study both subjects, that can distribute freely between them. Each grade is expressed in an scale from 0 to 10 points. The final grade is the average of the grades obtained in each of the subjects. For instance, if she gets 8 points in A and 6 points in B, then the final grade is $\frac{1}{2}(8+6) = 7$. She knows from previous experience that when devoting x hours to subject A she gets a grade of $10\frac{x}{4+x}$ points. In the same way, when she devotes y hours to study subject B, she gets $10\frac{y}{1+y}$ points.

(a) (5 points) Suppose that the student wants to maximize the average final grade, subject to the time constraint. Write down the associated Lagrange problem and find the optimal distribution of study time (x^*, y^*), as well as the multiplier.

(b) (5 points) Which is the maximum final grade? How much additional time should have been devoted to the study if she wanted to get 0.5 points more? Give and approximated but rigorous answer based on the interpretation of the multiplier.

Solution:

(a) The Lagrangian is $L(x, y, \lambda) = \frac{1}{2}(10\frac{x}{4+x} + 10\frac{y}{1+y}) + \lambda(25 - x - y)$, where λ is the multiplier associated to the constraint $x + y = 25$. The conditions of stationarity of the lagrangian are

$$\begin{cases} \frac{\partial L}{\partial x} &= \frac{20}{(4+x)^2} - \lambda = 0, \\ \frac{\partial L}{\partial y} &= \frac{5}{(1+y)^2} - \lambda = 0, \\ \frac{\partial L}{\partial \lambda} &= 25 - x - y = 0 \end{cases} .$$

Eliminating λ from the two first equations we have

$$\frac{4}{(4+x)^2} = \frac{1}{(1+y)^2} \Rightarrow 4(1+y)^2 = (4+x)^2 \Rightarrow 2(1+y) = 4 + x \Rightarrow x = 2y - 2.$$

Plugging this equality into the time constraint, we get $25 - x - y = 25 - (2y - 2) - y = 27 - 3y = 0$, hence $y = 9$ and then $x = 16$. Notice that the set $\{x + y = 25, x \geq 0, y \geq 0\}$ is a compact set. As the objective function is continuous on $\{x \geq 0, y \geq 0\}$, the Theorem of Weierstrass guarantees that a global maximum of the problem exists. It has to be the point $(x^*, y^*) = (16, 9)$, which is the only candidate (the global minimum is located at the extreme point $(25, 0)$; it is not found by using the Lagrange Theorem. Why?). Note that he multiplier is $\lambda = \frac{1}{20} = 0.05$.

(b) The maximum final grade is attained at the point found in part (a). We compute it is equal to

$$v^*(\text{time} = 25) = \frac{1}{2}\left(10\frac{x^*}{4+x^*} + 10\frac{y^*}{1+y^*}\right) = \frac{1}{2}\left(10\frac{16}{20} + 10\frac{9}{10}\right) = 5\frac{17}{10} = 8.5$$

points (not so bad!). To answer the final question we do not solve a new Lagrange problem. Instead, we resort to the interpretation of the multiplier as a shadow price or marginal increment of the optimal grade due to the increment of one unit of time devoted to study:

$$\Delta v^* = v^*(25 + h) - v^*(25) \approx h \times \lambda.$$

Notice that $v^*(25) = 8.5$ and $\lambda = 0.05$. We need to find h, the additional number of hours to reach a grade of $v^*(25 + h) = 9$. Using the identity above we obtain

$$9 - 8.5 \approx 0.05h \Rightarrow h = 10.$$

Hence, in this model, to get half a point more, the student has to do a considerable effort, devoting 10 additional hours to study.

$\boxed{5}$

Consider the functions $g(u,v) = \frac{1}{2}\ln(4u^2+v)$, $u = \frac{x-y}{2}$, $v = 2xy$ and $f(x,y) = g(u(x,y), v(x,y))$.

(a) (4 points) Compute the derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ by using the chain rule. Hint: the final answer has to contain expressions that depend only on x, y.

(b) (6 points) Compute the directional derivative of f at the point $(1,2)$ in the direction $(-\frac{2}{5}, \frac{1}{5})$. In other words, compute $D_{(-\frac{2}{5},\frac{1}{5})}f(1,2)$. Is $(-\frac{2}{5}, \frac{1}{5})$ the direction in which f decreases the fastest at the point $(1,2)$?

Solution:

(a) One has

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} = \frac{8u}{2(4u^2+v)}\frac{1}{2} + \frac{1}{2(4u^2+v)}2y = \frac{2u+y}{4u^2+v} = \frac{x}{x^2+y^2},$$
$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y} = \frac{8u}{2(4u^2+v)}\frac{-1}{2} + \frac{1}{2(4u^2+v)}2x = \frac{-2u+y}{4u^2+v} = \frac{y}{x^2+y^2}.$$

(b) Observe that $\nabla f(1,2) = \left(\frac{1}{5}, \frac{2}{5}\right)$. The direction of fastest decrease of f at the point $(1,2)$ is thus $-\nabla f(1,2) = -\left(\frac{1}{5}, \frac{2}{5}\right)$. On the other hand,

$$D_{(-\frac{2}{5},\frac{1}{5})}f(1,2) = \left(-\frac{2}{5}, \frac{1}{5}\right) \cdot \nabla f(1,2) = \left(-\frac{2}{5}, \frac{1}{5}\right) \cdot \left(\frac{1}{5}, \frac{2}{5}\right) = 0.$$

$\boxed{6}$

Consider the function

$$f(x,y) = \begin{cases} \frac{xy^3}{x^4+y^4}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) (5 points) Study the continuity of f in $(0,0)$. Is f differentiable at $(0,0)$?

(b) (5 points) Calculate the partial derivatives

$$\frac{\partial f}{\partial x}(0,0), \quad \frac{\partial f}{\partial y}(0,0)$$

and the directional derivative of f at $(0,0)$ along the direction $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. In other words, compute $D_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})}f(0,0)$.

Solution:

(a) f is not continuous at $(0,0)$, since$f(0,0) = 0$ but the directional limit through the line $y = x$ is

$$\lim_{\substack{(x,y)\to(0,0) \\ y=x}} \frac{xy^3}{x^4+y^4} = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}.$$

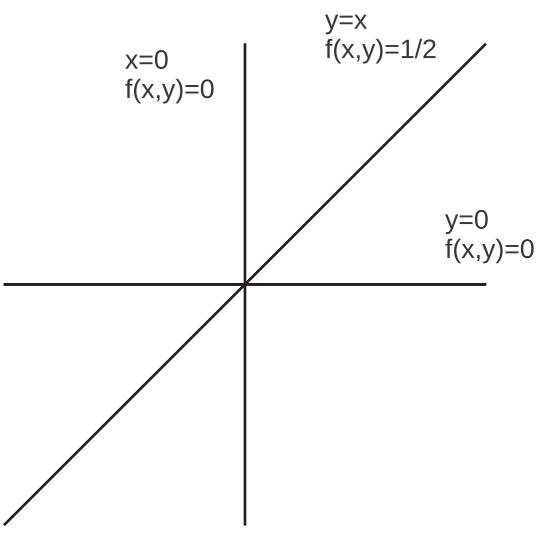
In consequence, f is not differentiable at $(0,0)$.

(b)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0,$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0.$$

$$D_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})}f(0,0) = \lim_{t\to 0} \frac{f(\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}})-f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^4}{4}}{\left(\frac{t^4}{4}+\frac{t^4}{4}\right)t} = \lim_{t\to 0} \frac{1}{2t},$$

does not exist.



Level curves in figure show f is not continious at $(0,0)$