University Carlos III of Madrid

Department of Economics
Mathematics II. Final Exam. June 2007.

IMPORTANT:

- **DURATION OF THE EXAM: 2h. 30min.**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam. These last two pages will not be graded. **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.
- Each part of the exam counts $0'5$ points.

Last Name: **Name:**

DNI: **Group:**

(1) *Consider the following system of linear equations,*

$$\begin{aligned} x + 3y + 2z &= 1 \\ 3x + y + 2z &= b \\ x + y + az &= 2b \end{aligned}$$

where $a, b \in \mathbb{R}$ are parameters.

(a) *Classify the system according to the values of the parameters a, b.*
(b) *Solve the above system for the values of $a = 1$ and $b = 1/7$.*

(a) We compute first the ranks of the (augmented) matrix associated to the system. For this, we will do elementary operations.

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 3 & 1 & 2 & b \\ 1 & 1 & a & 2b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -8 & -4 & -3+b \\ 0 & -2 & -2+a & -1+2b \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -2 & -2+a & -1+2b \\ 0 & -8 & -4 & -3+b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -2 & -2+a & -1+2b \\ 0 & 0 & 4-4a & 1-7b \end{array}\right)$$

The rank of A is 2 if and only if $a = 1$. When $a = 1$ The rank of the augmented matrix is 3, if $b \neq 1/7$ and 2 if $b = 1/7$. Thus, the system,

- has a unique solution if $a \neq 1$.
- is underdetermined if $a = 1$ and $b = 1/7$.
- is overdetermined if $a = 1$ and $b \neq 1/7$.

(b) Plugging the values $a = 1$ and $b = 1/7$, we see that the original system is equivalent to the following one

$$\begin{aligned} x + 3y + 2z &= 1 \\ -2y - z &= -5/7 \end{aligned}$$

Taking y as the parameter. The solution may be written as $x = y - 3/7$, $z = 5/7 - 2y$ with $y \in \mathbb{R}$.

(2) *Given the matrix*

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) *Find the characteristic polynomial and the eigenvalues.*
(b) *Find a basis of* $\mathbb{R}^3$ *consisting of eigenvectors of* A.
(c) *Compute* A^{10}. *(You may use that* $2^{10} = 1024$.*)*

(a) The characteristic polynomial is $-(\lambda+1)(\lambda-1)(\lambda-2)$. The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$ y $\lambda_3 = 2$

(b) It is easy to compute that

$$\begin{aligned} S(-1) &=< (-1,1,0) > \\ S(1) &=< (0,-1,1) > \\ S(2) &=< (1,0,0) > \end{aligned}$$

Hence, the diagonal form D and the matrix change of basis P are

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(c) Note that $A = PDP^{-1}$ so

$$A^{10} = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1024 \end{pmatrix} P^{-1}$$

Since,

$$P^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

we have that

$$A^{10} = \begin{pmatrix} 1024 & 1023 & 1023 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(3) *Given the linear map* $f : \mathbb{R}^3 \to \mathbb{R}^4$,

$$f(x, y, z) = (2x - y + z, x - y, 3x - 2y + z, y + z)$$

(a) *Compute the matrix of* f *with respect to the canonical bases.*
(b) *Compute the dimensions of the kernel and the image and a set of equations for these subspaces.*
(c) *Find a basis of the image of* f *and a basis of the kernel of* f.

(a) The matrix A associated to f with respect to the canonical bases is

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & 0 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(b) We compute its reduced form,

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & x \\ 1 & -1 & 0 & y \\ 3 & -2 & 1 & z \\ 0 & 1 & 1 & t \end{array}\right) \sim \begin{pmatrix} 1 & -1 & 0 & y \\ 0 & 1 & 1 & t \\ 3 & -2 & 1 & z \\ 2 & -1 & 1 & x \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & y \\ 0 & 1 & 1 & t \\ 0 & 1 & 1 & -3y + z \\ 0 & 1 & 1 & x - 2y \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & y \\ 0 & 1 & 1 & t \\ 0 & 0 & 0 & -t - 3y + z \\ 0 & 0 & 0 & x + y - z \end{pmatrix}$$

From the above, we see that $\dim \operatorname{Im}(f) = \operatorname{rank}(A) = 2$ and a system of linear equations that define $\operatorname{Im}(f)$ is

$$\begin{aligned} -3y - t + z &= 0 \\ x + y - z &= 0 \end{aligned}$$

From the formula

$$3 = \dim(\operatorname{Im}(f)) + \dim(\ker(f))$$

we get that $\dim(\ker(f)) = 1$ nd a system of linear equations that determine $\ker(f)$ is

$$\begin{aligned} x - y &= 0 \\ y + z &= 0 \end{aligned}$$

(c) We may obtain basis of $\operatorname{Im}(f)$ starting from the columns of A, $\{(2, 1, 3, 0), (1, 0, 1, 1)\}$. To compute a basis of $\ker(f)$ we solve the above system of linear equations

$$\begin{aligned} x - y &= 0 \\ y + z &= 0 \end{aligned}$$

We choose y as the parameter. The dependent variables are x and z and they are determined by the equations $x = y$, $z = -y$. Thus,

$$\ker(f) = \{(x, y, z) \in \mathbb{R}^3 : x = y, z = -y\} = \{(y, y, -y) : y \in \mathbb{R}\}$$

and a basis of $\ker(f)$ is $\{(1, 1, -1)\}$.

(4) *Given the set*

$$A = \{(x, y) \in \mathbb{R}^2 : \quad x - 2y \geq -6, \quad x \leq 0, \quad y \geq 0\}$$

(a) *Draw the set A, computing its intersection with the axes. Draw its boundary and interior and discuss whether the set A is open, closed, bounded, compact and/or convex. You must explain your answer.*

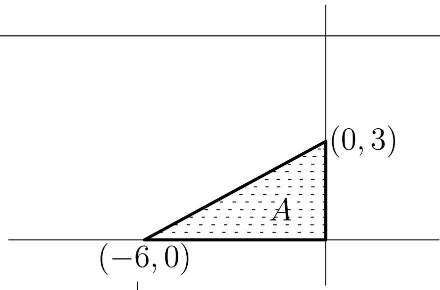
(b) *Show that the function*

$$f(x, y) = \frac{x^2}{(x+4)^2 + (y-2)^2}$$

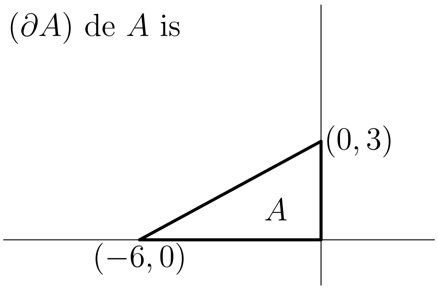
attains a maximum and a minimum on the set A.

(c) *Draw the level curves of the function $g(x, y) = y + x^2$ and use them to determine the maxima and minima of g on A.*

(a) The set A is



(b) The boundary (∂A) de A is

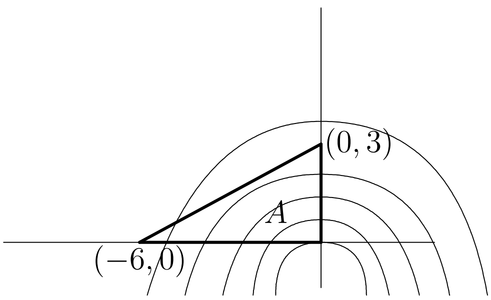


The interior of A is $A \setminus \partial A$, and the closure of A is $\bar{A} = A \cup \partial A = A$ (since $\partial A \subset A$). Therefore, A is closed, it is not open (since $\partial A \cap A \neq \emptyset$), is compact (closed and bounded). Finally, the set A is convex.

We may also show that A is closed and convex as follows: The functions $h_1(x, y) = x - 2y + 6$, $h_2(x, y) = x$ and $h_3(x, y) = y$ are continuous and linear. Hence, the set $A = \{(x, y) \in \mathbb{R}^2 : h_1(x, y) \geq 0, h_2(x, y) \leq 0, \quad h_3(x, y) \geq 0\}$ is closed and convex.

(b) The function f is continuous except at the point $(-4, 2) \notin A$. Therefore, f is continuous in A, which is compact. By Weierstrass' Theorem, la function attains a maximum and a minimum on the set A.

(c) The level curves of g are parabolas of the form $y = C - x^2$ con $C \in \mathbb{R}$.



Graphically, we see that the minimum is attained at the point $(0, 0)$ and el maximum at the point $(-6, 0)$.

(5) *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{x^2+x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) *Study if the function* f *is continuous at the point* $(0,0)$. *Study at which points of* $\mathbb{R}^2$ *the function* f *is continuous.*
(b) *Compute the partial derivatives of* f *at the point* $(0,0)$.
(c) *At which points of* $\mathbb{R}^2$ *is the function* f *differentiable?*

(a) We study the limit when $(x,y) \to (0,0)$ using straight lines $x(t) = t$, $y(t) = kt$ with $k \in \mathbb{R}$,

$$\lim_{t\to 0} f(t,kt) = \lim_{t\to 0} \frac{t^2(1+kt)}{t^2+k^2t^2} = \lim_{t\to 0} \frac{(1+kt)}{1+k^2} = \frac{1}{1+k^2}$$

and since it depends on the parameter $k \in \mathbb{R}$, the limit does not exist and the function is not continuous.

(b) The partial derivatives of f at the point $(0,0)$ are

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t}$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t}$$

Note that for every $t \neq 0$,

$$f(t,0) = \frac{t^2}{t^2} = 1$$
$$f(0,t) = 0$$

Therefore,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{1}{t} \quad \text{does not exist}$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{0}{t} = 0$$

(c) For each $(x,y) \neq (0,0)$, the function $f(x,y)$ is defined as a quotient of polynomials and the denominator does not vanish. Therefore, for $(x,y) \neq (0,0)$, all the partial derivatives exist and are continuous. We conclude that the function is differentiable at every point $(x,y) \neq (0,0)$.

At the point $(0,0)$ the function is not continuous and, hence, is not differentiable. (Alternatively, one may argue that

$$\frac{\partial f}{\partial x}(0,0)$$

does not exist.)

(6) *Consider the function*

$$f(x,y) = 2x^4 + y^4 - 2x^2 - 2y^2$$

(a) *Determine the critical points of f.*
(b) *Classify the critical points of f that you obtained in part (a).*
(c) *Determine the largest open set A of $\mathbb{R}^2$ where the function f is concave.*
(d) *Find the global extreme points of f on A.*

(a) The partial derivatives of the function are

$$\frac{\partial f}{\partial x} = 8x^3 - 4x$$
$$\frac{\partial f}{\partial y} = 4y^3 - 4y$$

and, since the function is differentiable in all of $\mathbb{R}^2$, the critical points are solutions of the system of equations,

$$8x^3 - 4x = 0$$
$$4y^3 - 4y = 0$$

The solutions of the first equation are

$$x = 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$$

and the solutions of the second equation are

$$y = 0, 1, -1$$

From here we obtain 9 critical points:

$$(0,0), (0,\pm 1), (\pm\frac{1}{\sqrt{2}}, 0), (\pm\frac{1}{\sqrt{2}}, \pm 1),$$

(b) The Hessian matrix of f is

$$\mathrm{H}\, f(x,y = \begin{pmatrix} 24x^2 - 4 & 0 \\ 0 & 12y^2 - 4 \end{pmatrix}$$

and the eigenvalues are $\lambda_1 = 24x^2 - 4$ and $\lambda_2 = 12y^2 - 4$ We see that

$$\mathrm{H}\, f(0,0) = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} \qquad \mathrm{H}\, f(\pm\frac{1}{\sqrt{2}}, \pm 1) = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

$$\mathrm{H}\, f(0,\pm 1) = \begin{pmatrix} -4 & 0 \\ 0 & 8 \end{pmatrix} \qquad \mathrm{H}\, f(\pm\frac{1}{\sqrt{2}}, 0) = \begin{pmatrix} 8 & 0 \\ 0 & -4 \end{pmatrix}$$

so

$$(0,0) \qquad \text{is a local maximum}$$

the four points

$$(\pm\frac{1}{\sqrt{2}}, \pm 1) \qquad \text{are local minima}$$

and the four points

$$(0,\pm 1), \quad (\pm\frac{1}{\sqrt{2}}, 0) \qquad \text{are saddle points}$$

(c) f is of class $C^2(\mathbb{R}^2)$, so f is concave if and only if $\mathrm{H}\, f(x,y)$ is negative definite or negative semidefinite. This happens if $24x^2 - 4 \leq 0$ and $12y^2 - 4 \leq 0$. So we get that

$$x^2 \leq \frac{1}{6} \quad \text{e} \quad y^2 \leq \frac{1}{3}$$

that is,

$$-\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}} \quad \text{e} \quad -\frac{1}{\sqrt{3}} < y < \frac{1}{\sqrt{3}}$$

Hence, the largest open set where f is concave is

$$A = \{(x,y) \in \mathbb{R}^2 : -\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}}, \quad -\frac{1}{\sqrt{3}} < y < \frac{1}{\sqrt{3}}\}$$

(d) The set A is convex. In the set A, the Hessian matrix, $\mathrm{H}\, f(x,y)$, is negative definite. Hence, f is strictly concave in A and its unique critical point in that set, $(0,0)$ is the global maximum in A.
We study now if there is global a minimum on A. Since the set A is open, if there were a minimum of f in A, it would be a critical point of f. But, since f is concave, all the critical points of f are global maxima. Therefore, f does not have a minimum (local or global) in A.

(7) *Consider the function*

$$f(x, y, z) = 2x^2 + y^2 - x - z + 4z^2$$

and the set

$$A = \{(x, y, z) : x + y = z\}$$

(a) *Find the Lagrange equations that determine the extreme points of f in the set A.*
(b) *Determine the points that satisfy the Lagrange equations and find the extreme points of f, specifying whether they correspond to a maximum or minimum.*

(a) The Lagrangian function is

$$L(x, y, z, \lambda) = 2x^2 + y^2 - x - z + 4z^2 + \lambda(x + y - z).$$

The Lagrange equations are :

$$\begin{aligned} \frac{\partial L}{\partial x} &= 4x - 1 + \lambda = 0 \\ \frac{\partial L}{\partial y} &= 2y + \lambda = 0 \\ \frac{\partial L}{\partial z} &= -1 + 8z - \lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= x + y - z = 0 \end{aligned}$$

(b) From the first equations, we see that $4x - 1 = 2y = 1 - 8z$. Substituting this in the last equation, we get that

$$x = \frac{3}{14}, \quad y = -\frac{1}{14}, \quad z = \frac{1}{7}, \quad \lambda = \frac{1}{7}.$$

From this we see that the unique point which satisfies the Lagrange equations is $\left(\frac{3}{14}, -\frac{1}{14}, \frac{1}{7}\right)$. The Hessian matrix of the function L is

$$HL(x, y, z) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

which is positive definite. Hence, $\left(\frac{3}{14}, -\frac{1}{14}, \frac{1}{7}\right)$ is a minimum. since the Lagrangian function does not have any other critical points, there is no maximum.