University Carlos III
Department of Economics
Mathematics II. Final Exam. May 31st 2021

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Given the following system of linear equations,

$$\left\{\begin{array}{rcl} ax+y+z & = & b \\ ax+y+az & = & a \\ x+ay+z & = & 1 \end{array}\right.$$

where $a, b \in \mathbb{R}$ are parameters.

(a) Classify the system according to the values of a and b. $\boxed{\textbf{10 points}}$

Solution: *The matrix associated with the system is*

$$\begin{pmatrix} a & 1 & 1 & b \\ a & 1 & a & a \\ 1 & a & 1 & 1 \end{pmatrix}$$

Rearranging the rows, we obtain

$$\begin{pmatrix} 1 & a & 1 & 1 \\ a & 1 & a & a \\ a & 1 & 1 & b \end{pmatrix}$$

Next, we perform the following operations

$$\text{row } 2 \mapsto \text{row } 2 - a \times \text{row } 1$$

$$\text{row } 3 \mapsto \text{row } 3 - a \times \text{row } 1$$

And we obtain that the original system is equivalent to another one whose augmented matrix is the following

$$\begin{pmatrix} 1 & a & 1 & 1 \\ 0 & 1-a^2 & 0 & 0 \\ 0 & 1-a^2 & 1-a & b-a \end{pmatrix}$$

Now, we perform the operation row 3 $\mapsto$ row 3$-$ row 2 and we obtain

$$\begin{pmatrix} 1 & a & 1 & 1 \\ 0 & 1-a^2 & 0 & 0 \\ 0 & 0 & 1-a & b-a \end{pmatrix}$$

We see that

(i) *if $a^2 \neq 1$, then* $\operatorname{rank} A = \operatorname{rank}(A|b) = 3$. *The system is consistent with a unique solution.*

(ii) *If $a = 1$, the system is consistent if and only if $b = 1$. In the latter case,* $\operatorname{rank} A = \operatorname{rank}(A|b) = 1$. *The system is underdetermined with two parameters.*

(iii) *If $a = -1$, then* $\operatorname{rank} A = \operatorname{rank}(A|b) = 2$. *The system is underdetermined with one parameter.*

(b) Solve the above system when $a = -1$. $\boxed{\textbf{10 points}}$

Solution: *The proposed system of linear equations is equivalent to the following one*

$$\left\{\begin{array}{lcr} x-y+z & = & 1 \\ 2z & = & b+1 \end{array}\right.$$

The solution is

$$x \in \mathbb{R}, \quad y = x + \frac{b-1}{2}, \quad z = \frac{b+1}{2}$$

(2) Consider the set

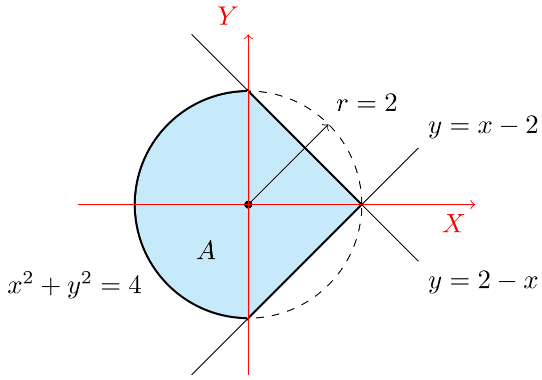
$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4, \;\; y \leq 2 - x, \;\; y \geq x - 2\}$$

and the function

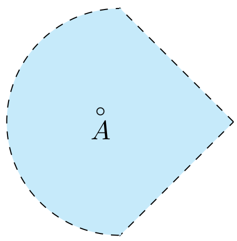
$$f(x,y) = -\ln\left((x+1)^2 + y^2\right)$$

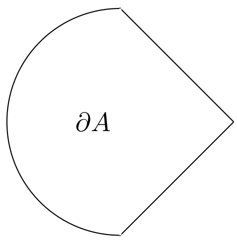
(a) Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded, compact or convex. $\boxed{\textbf{10 points}}$

Solution: *The set A is approximately as indicated (in blue) in the picture.*



The interior and the boundary are





The set A closed because $\partial A \subset A$. It is not open because $A \cap \partial A \neq \emptyset$. It is bounded. Therefore, the set A is compact. It is convex because the set A is the intersection of three convex sets $A = B \cap C \cap D$ with

$$B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

$$C = \{(x,y) \in \mathbb{R}^2 : y \leq x - 2\}$$

and

$$D = \{(x,y) \in \mathbb{R}^2 : y \geq 2 - x\}$$

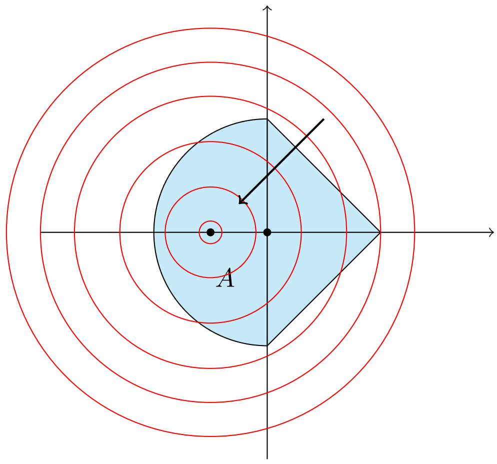
The function $g(x) = x^2 + y^2$ is convex. Therefore B is convex. The sets C and D are half-planes and, hence, also convex. Since A is the intersection of convex sets, it is also convex.

(b) State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A. Using the level curves, determine (if they exist) the extreme global points of f on the set A. $\boxed{\textbf{10 points}}$

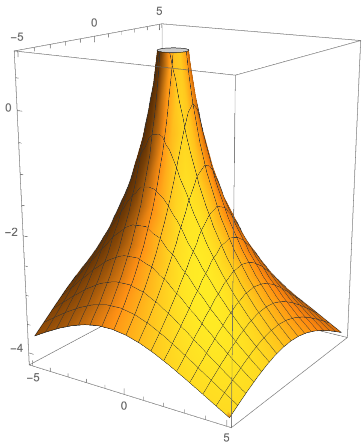
Solution: *The set A is compact. The function $f(x,y) = -\ln\left((x+1)^2 + y^2\right)$ is continuous in its domain of definition $D(f) = \mathbb{R}^2 \setminus \{(-1,0)\}$. Since, $(-1,0) \in A$, the function f is not continuous in all of A. Weierstrass' Theorem does not apply. Further,*

$$\lim_{(x,y)\to(-1,0)} f(x,y) = +\infty$$

We conclude that the function does not attain a maximum value in A. The level curves are given by the equation $(x+1)^2 + y^2 = c$, $c > 0$. So, the level curves of f are circles of center $(-1,0)$ and radius $\sqrt{c}$. In the picture we represent the level curves in red color. The arrow represents the direction of growth of the function f



Graphically, we see that the minimum value is attained at the point $(2,0)$*. To help in understanding the exercise, the graph of the function* f *is*



The vertical asymptote occurs at the point $(-1,0)$*.*

(3) Consider the function $f(x, y) = x^2 + 4xy - 2x + 2y^3 + 6y^2 - 20y$.

(a) Determine the largest open subset of $\mathbb{R}^2$ where the function f is strictly concave or convex. **10 points**

Solution: *The gradient of the function f is*

$$\nabla f(x, y) = \big(2x + 4y - 2, 4x + 6y^2 + 12y - 20\big)$$

We obtain now the Hessian matrix

$$\mathrm{H}\, f(x, y) = \begin{pmatrix} 2 & 4 \\ 4 & 12y + 12 \end{pmatrix}$$

We have $D_1 = 2 > 0$, $D_2 = 8 + 24y$. So, $D_2 > 0$ if and only if $y > -1/3$. Hence, the function is convex in the set $\{(x, y) \in \mathbb{R}^2 : y > -1/3\}$. The function is not concave in any open subset of $\mathbb{R}^2$.

(b) Find the critical points of f. Classify the critical points into (local/global) maxima/minima or saddle points. **10 points**

Solution: *The critical points satisfy the equations*

$$2x + 4y - 2 = 0, \quad 4x + 6y^2 + 12y - 20 = 0$$

The solutions are

$$x = 5, y = -2, \quad x = -\frac{5}{3}, y = \frac{4}{3}$$

The Hessian matrix evaluated at the point $x = 5$,$y = -2$ is

$$\mathrm{H}\, f(5, -2) = \begin{pmatrix} 2 & 4 \\ 4 & -12 \end{pmatrix}$$

The principal dominant minors of the Hessian matrix are $D_1 = 2 > 0$, $D_2 = -40 < 0$. Thus, the point $(5, -2)$ corresponds to saddle point. The Hessian matrix evaluated at the point $x = -\frac{5}{3}$, $y = \frac{4}{3}$ is

$$\mathrm{H}\, f\left(-\frac{5}{3}, \frac{4}{3}\right) = \begin{pmatrix} 2 & 4 \\ 4 & 28 \end{pmatrix}$$

The principal dominant minors of the Hessian matrix are $D_1 = 2 > 0$, $D_2 = 40 > 0$. Thus, the point $\left(-\frac{5}{3}, \frac{4}{3}\right)$ corresponds to a local minimum. It does not correspond to a global minimum because,

$$\lim_{y \to -\infty} f(0, y) = \lim_{y \to -\infty} \big(2y^3 + 6y^2 - 20y\big) = -\infty$$

(4) Consider the set of equations

$$\begin{aligned} xz + yz &= -4 \\ x + y^2 - z &= -2 \end{aligned}$$

(a) Prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x, y, z) = (-1, -1, 2)$. $\boxed{\textbf{10 points}}$

Solution: *We first remark that $(x, y, z) = (-1, -1, 2)$ is a solution of the system of equations. The functions $f_1(x, y, z) = xz + yz + a$ and $f_2(x, y, z) = x + y^2 - z + 2$ are of class C^∞. We compute*

$$\begin{vmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{vmatrix}_{(x,y,z)=(-1,-1,2)} = \begin{vmatrix} z & x+y \\ 2y & -1 \end{vmatrix}_{(x,y,z)=(-1,-1,2)} = \begin{vmatrix} 2 & -2 \\ -2 & -1 \end{vmatrix} = -6$$

By the implicit function theorem, the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x, y, z) = (-1, -1, 2)$.

(b) Compute

$$y'(-1), z'(-1)$$

and the first order Taylor polynomial of $y(x)$ and $z(x)$ at the point $x_0 = -1$. $\boxed{\textbf{10 points}}$

Solution: *Differentiating implicitly with respect to x,*

$$\begin{aligned} zy' + yz' + xz' + z &= 0 \\ 2yy' - z' + 1 &= 0 \end{aligned}$$

We plug in the values $(x, y, z) = (-1, -1, 2)$ to obtain the following

$$\begin{aligned} 2y'(-1) - 2z'(-1) + 2 &= 0 \\ 1 - 2y'(-1) - z'(-1) &= 0 \end{aligned}$$

So,

$$y'(-1) = 0, \quad z'(-1) = 1$$

Thus, Taylor's polynomial of order 1 of the function $y(x)$ at the point $x_0 = -1$ is

$$P_1(x) = y(-1) + y'(-1)(x + 1) = -1$$

and Taylor's polynomial of order 1 of the function $z(x)$ at the point $x_0 = -1$ is

$$Q_1(x) = z(-1) + z'(-1)(x + 1) = x + 3$$

(5) Consider the extreme points of the function

$$f(x,y) = 2x^3 - y^3$$

in the set

$$S = \{(x,y) : x^2 + y^2 = 5\}$$

(a) Write the Lagrangian function and the Lagrange equations and their solutions. **10 points**

Solution: *The Lagrangian is*

$$\mathcal{L}(x,y) = 2x^3 - y^3 - \lambda\left(x^2 + y^2 - 5\right)$$

The Lagrange equations are

$$\begin{aligned} 6x^2 - 2\lambda x &= 0 \\ -3y^2 - 2\lambda y &= 0 \\ x^2 + y^2 &= 5 \end{aligned}$$

which may be written as

$$\begin{aligned} 2x(3x - \lambda) &= 0 \\ y(3y + 2\lambda) &= 0 \\ x^2 + y^2 &= 5 \end{aligned}$$

We obtain immediately the solutions

$$\begin{aligned} x = 0, \quad y = -\sqrt{5}, \quad \lambda = \frac{3\sqrt{5}}{2} \\ x = 0, \quad y = \sqrt{5}, \quad \lambda = -\frac{3\sqrt{5}}{2} \\ x = -\sqrt{5}, \quad y = 0, \quad \lambda = -3\sqrt{5} \\ x = \sqrt{5}, \quad y = 0, \quad \lambda = 3\sqrt{5} \end{aligned}$$

Otherwise, we obtain the equations

$$\begin{aligned} 3x - \lambda &= 0 \\ 3y + 2\lambda &= 0 \\ x^2 + y^2 &= 5 \end{aligned}$$

That is, $\lambda = 3x$ *and substituting in the second equation, we obtain* $3y + 6x = 0$*. Hence* $y = -2x$ *and substituting in the third equation we obtain* $5x^2 = 5$*. Thus, we obtain the additional solutions*

$$\begin{aligned} x = -1, \quad y = 2, \quad \lambda = -3 \\ x = 1, \quad y = -2, \quad \lambda = 3 \end{aligned}$$

(b) Find the global maximum and minimum values of f on the set S. **10 points**

Solution: *The set* S *is compact and the function* f *is continuous. Hence the the function* f *attains a global maximum and minimum in* S*, by Weiestrass' Theorem. Note that*

$$\begin{aligned} f(-1,2) &= -10 \\ f\left(0,-\sqrt{5}\right) &= 5\sqrt{5} \\ f\left(0,\sqrt{5}\right) &= -5\sqrt{5} \\ f(1,-2) &= 10 \\ f\left(-\sqrt{5},0\right) &= -10\sqrt{5} \\ f\left(\sqrt{5},0\right) &= 10\sqrt{5} \end{aligned}$$

We see that the global maximum is attained at the point $(\sqrt{5},0)$*. The maximum value is* $f(\sqrt{5},0) = 10\sqrt{5}$*. The global minimum is attained at the point* $(-\sqrt{5},0)$*. The minimum value is* $f(-\sqrt{5},0) = -10\sqrt{5}$*. Of course, these are also local extreme points.*