(1) Given the following system of linear equations,

$$\begin{cases} 3x - y + 2z &= 1 \\ x + 4y + z &= b \\ 2x - 5y + (a+1)z &= 0 \end{cases}$$

where $a, b \in \mathbb{R}$ are parameters.

(a) Classify the system according to the values of a and b. **5 points**

Solution: *The matrix associated with the system is*

$$\begin{pmatrix} 3 & -1 & 2 & 1 \\ 1 & 4 & 1 & b \\ 2 & -5 & a+1 & 0 \end{pmatrix}$$

Exchanging rows 1 and 2 we obtain

$$\begin{pmatrix} 1 & 4 & 1 & b \\ 3 & -1 & 2 & 1 \\ 2 & -5 & a+1 & 0 \end{pmatrix}$$

Next, we perform the following operations

$$\text{row } 2 \mapsto \text{row } 2 - 3 \times \text{row } 1$$

$$\text{row } 3 \mapsto \text{row } 3 - 2 \times \text{row } 1$$

And we obtain that the original system is equivalent to another one whose augmented matrix is the following

$$\begin{pmatrix} 1 & 4 & 1 & b \\ 0 & -13 & -1 & 1-3b \\ 0 & -13 & a-1 & -2b \end{pmatrix}$$

Now, we perform the operation row 3 $\mapsto$ row 3 $-$ row 2 and we obtain

$$\begin{pmatrix} 1 & 4 & 1 & b \\ 0 & -13 & -1 & 1-3b \\ 0 & 0 & a & b-1 \end{pmatrix}$$

We see that

(i) *if $a \neq 0$, then $\operatorname{rank} A = \operatorname{rank}(A|b) = 3$. The system is consistent with a unique solution.*

(ii) *If $a = 0$ the system is consistent if and only if $b = 1$. In the latter case, $\operatorname{rank} A = \operatorname{rank}(A|b) = 2$. The system is underdetermined with one parameter.*

(b) Solve the above system for all the values a and b for which the system is consistent. **5 points**

Solution: *Suppose first that $a \neq 0$. The proposed system of linear equations is equivalent to the following one*

$$\begin{cases} x + 4y + z &= b \\ -13y - z &= 1 - 3b \\ az &= b - 1 \end{cases}$$

The solution is

$$x = \frac{a(b+4) - 9b + 9}{13a}, \quad y = \frac{3ab - a - b + 1}{13a}, \quad z = \frac{b-1}{a}$$

Suppose now that $a = 0$, $b = 1$. The proposed system of linear equations is equivalent to the following one

$$\begin{cases} x + 4y + z &= 1 \\ -13y - z &= -2 \end{cases}$$

The solution is

$$x = 9y - 1, \quad z = 2 - 13y, \quad y \in \mathbb{R}$$

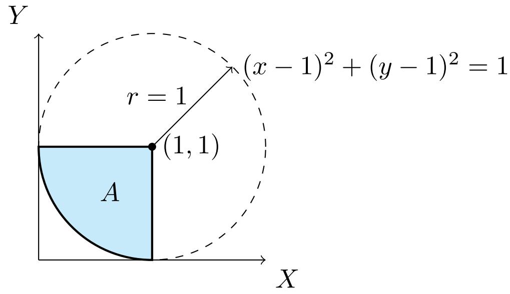
(2) Consider the set $A = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le 1, (x-1)^2 + (y-1)^2 \le 1\}$ and the function

$$f(x,y) = x - y$$

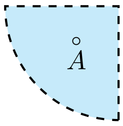
defined on A.

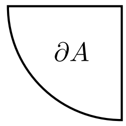
(a) Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded, compact or convex. **5 points**

Solution: *The set A is approximately as indicated (in blue) in the picture.*



The interior and the boundary are





The set A closed because $\partial A \subset A$. It is not open because $A \cap \partial A \neq \emptyset$. It is bounded. Therefore, the set A is compact. It is convex because the set A is the intersection of three sets $A = B \cap C \cap D$ with

$$B = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 \le 1\}$$
$$C = \{(x,y) \in \mathbb{R}^2 : x \le 1\}$$

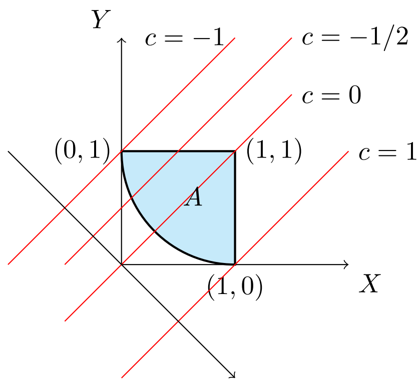
and

$$D = \{(x,y) \in \mathbb{R}^2 : y \le 1\}$$

The function $g(x) = (x-1)^2 + (y-1)^2$ is convex. Therefore B is convex. The sets C and D are half-planes and, hence, also convex. Since A is the intersection of convex sets, it is also convex.

(b) State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A. Using the level curves, determine (if they exist) the extreme global points of f on the set A. **5 points**

Solution: *The function $f(x,y) = x - y$ is continuous and the set A is compact. Weierstrass' Theorem may be applied. The function f attains a global maximum and a global minimum on A. The level curves are of the form $y = x - c$, $c \in \mathbb{R}$. In the picture we represent the level curves in red color. The arrow represents the direction of growth of the function f*



Graphically, we see that the maximum value is attained at the point $(1,0)$ and the minimum value is attained at the point $(0,1)$.

(3) Consider the function $f(x, y) = ax^3 + ay^2 - 2xy$ where $a \in \mathbb{R}$ is a parameter and $a > 0$.

(a) Compute the gradient and the Hessian matrix of the function f. Compute the Taylor polynomial of degree 2 of f, centered at the point $x = 0$, $y = 1$. Compute the critical points of f. **5 points**

Solution: *The gradient of the function is*

$$\nabla f(x, y) = \big(3ax^2 - 2y, 2ay - 2x\big)$$

We obtain now the Hessian matrix

$$Hf(x, y) \begin{pmatrix} 6ax & -2 \\ -2 & 2a \end{pmatrix}$$

Note that $f(0, 1) = a$. *The gradient evaluated at the point* $x = 0$, $y = 1$ *is*

$$\nabla f(0, 1) = (-2, 2a)$$

The Hessian matrix evaluated at the point $x = 0$, $y = 1$ *is*

$$Hf(0, 1) = \begin{pmatrix} 0 & -2 \\ -2 & 2a \end{pmatrix}$$

and Taylor's polynomial of degree 2 of f *centered at the point* $x = 0$, $y = 1$ *is*

$$P_2 = a - 2x + 2a(y - 1) + \frac{1}{2}(2a(y - 1)^2 - 4x(y - 1)) = ay^2 - 2xy$$

The critical points satisfy the equations

$$3ax^2 - 2y = 0, \quad ay - x = 0$$

The solutions are

$$x = 0, \quad y = 0$$

and

$$x = \frac{2}{3a^2}, \quad y = \frac{2}{3a^3}$$

(b) Determine the largest open set of $\mathbb{R}^2$ where the function f is concave or convex, depending on the values of the parameter a. **5 points**

Solution: *The principal dominant minors of the Hessian matrix are*

$$D_1 = 6ax, \quad D_2 = 12a^2x - 4$$

If $x > \frac{1}{3a^2}$, *then* $D_1, D_2 > 0$. *Therefore, the function is strictly convex in the set*

$$\{(x, y) \in \mathbb{R}^2 : x > \frac{1}{3a^2}\}$$

On the other hand, f *cannot be concave, because* $D_1 < 0$ *if and only if* $x < 0$. *But, then* $D_2 < 0$ *and the associated quadratic form is indefinite.*

(4) Consider the set of equations

$$\begin{aligned} xy + zy^2 &= 1 \\ x + y - z &= 0 \end{aligned}$$

(a) Prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x, y, z) = (0, 1, 1)$. **4 points**

Solution: *We first remark that $(x, y, z) = (0, 1, 1)$ is a solution of the system of equations. The functions $f_1(x, y, z) = xy + zy^2$ and $f_2(x, y, z) = x + y - z$ are of class C^∞. We compute*

$$\begin{vmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{vmatrix}_{(x,y,z)=(0,1,1)} = \begin{vmatrix} x + 2zy & y^2 \\ 1 & -1 \end{vmatrix}_{(x,y,z)=(0,1,1)} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

By the implicit function theorem, the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x, y, z) = (0, 1, 1)$.

(b) Compute

$$y'(0), z'(0)$$

and the first order Taylor polynomial of $y(x)$ and $z(x)$ at the point $x_0 = 0$. **6 points**

Solution: *Differentiating implicitly with respect to x,*

$$\begin{aligned} y + xy' + z'y^2 + 2zyy' &= 0 \\ 1 + y' - z' &= 0 \end{aligned}$$

We plug in the values $(x, y, z) = (0, 1, 1)$ to obtain the following

$$\begin{aligned} 1 + z'(0) + 2y'(0) &= 0 \\ 1 + y'(0) - z'(0) &= 0 \end{aligned}$$

So,

$$z'(0) = \frac{1}{3}, \quad y'(0) = -\frac{2}{3}$$

Thus, Taylor's polynomial of order 1 of the function $y(x)$ at the point $x_0 = 0$ is

$$P_1(x) = y(0) + y'(0)x = 1 - \frac{2}{3}x$$

and Taylor's polynomial of order 1 of the function $z(x)$ at the point $x_0 = 0$ is

$$Q_1(x) = z(0) + z'(0)x = 1 + \frac{1}{3}x$$

(5) Consider the function

$$f(x, y) = xy - x^3 - y^2$$

(a) Determine the extreme points of f in the set $\mathbb{R}^2$. **5 points**

(b) Classify the extreme points of f in the set $\mathbb{R}^2$ and justify if they are local or global extreme points. **5 points**

Solution: *The function f is of class C^∞ in $\mathbb{R}^2$. We compute the gradient of f*

$$\nabla f(x, y) = (y - 3x^2, x - 2y)$$

The critical points are the solutions of the system of equations

$$y - 3x^2 = 0, \quad x - 2y = 0$$

The solutions are

$$(0, 0) \quad \textit{and} \quad \left(\frac{1}{6}, \frac{1}{12}\right)$$

The Hessian matrix is

$$Hf(x, y) = \begin{pmatrix} -6x & 1 \\ 1 & -2 \end{pmatrix}$$

At the point $(0, 0)$ we obtain

$$Hf(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

So, $D_2 = -1 < 0$. The associated quadratic form is indefinite. The point $(0, 0)$ is a saddle point.
At the point $\left(\frac{1}{6}, \frac{1}{12}\right)$ we obtain

$$Hf\left(\frac{1}{6}, \frac{1}{12}\right) = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$$

So, $D_1 = -1 < 0$, $D_2 = 1 > 0$, which is negative definite. Hence, the point $\left(\frac{1}{6}, \frac{1}{12}\right)$ corresponds to a local maximum. Since,

$$\lim_{x \to \infty} f(x, 0) = -\infty, \qquad \lim_{x \to -\infty} f(x, 0) = \infty$$

the point $\left(\frac{1}{6}, \frac{1}{12}\right)$ does not correspond to global maximum.

(6) Consider the function

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$$

and the set

$$S = \{(x, y, z) : x + y + z = 1\}$$

(a) Write the Lagrangian function and the Lagrange equations. **5 points**

(b) Using the Lagrangian method, find the minima of $f(x, y, z)$ on the set S. **5 points**

Solution: *The candidates for the global extreme points must satisfy the first-order necessary conditions. The Lagrangian is*

$$\mathcal{L}(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2 - \lambda(x + y + z - 1)$$

The first-order necessary conditions are:

$$\mathcal{L}_x(x, y, z) = \lambda + 2(x-1) = 0 \quad (1)$$

$$\mathcal{L}_y(x, y, z) = \lambda + 2(y-2) = 0 \quad (2)$$

$$\mathcal{L}_z(x, y, z) = \lambda + 2(z-3) = 0 \quad (3)$$

$$x + y + z = 1 \quad (4)$$

The solution is

$$x = -\frac{2}{3}, \quad y = \frac{1}{3}, \quad z = \frac{4}{3}, \quad \lambda = \frac{10}{3}$$

Note that the Lagrangian associated with the Hessian is

$$HL(x, y, z; \lambda) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

which is positive definite. Hence, the critical point corresponds to a local and global minimum.