Universidad Carlos III de Madrid

Department of Economics
Final Exam. Mathematics II. September 2003.

IMPORTANT:

- **DURATION OF THE EXAM: 2h. 30min.**
- Calculators are not allowed.
- **Hand in only this booklet.** Do not hand in any scratch paper. All your answers should be written in this booklet.
- You must identify yourself to the professor.
- Each part of the exam counts $0'5$ points.

Last Name(s):		**Name:**
DNI:	**Titulación:**	**Group:**

(1) The function $f(x, y) = 3x^2 + e^{xy}$ represents the profit of a firm that produces x units of good 1 and y units of good 2.
  (a) Find the gradient of f at the point $(1, 0)$.
  (b) Last year the company produced 1 unit of good 1 and 0 units of good 2. This year the company is forced to produce $(1 + \Delta x, \Delta y)$. Knowing that Δx and Δy are very small and that the company chooses them so that the profit increases the fastest, compute approximately

$$\frac{\Delta x}{\Delta y}$$

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(2) Given the linear mapping $f : \mathbb{R}^4 \to \mathbb{R}^3$,

$$f(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3, 2x_1 + 2x_2 + 2x_3, x_1 + x_2 + 3x_3 + x_4)$$

  (a) Find the matrix of f with respect to the canonical bases and the dimension of the kernel and the image.
  (b) Find a basis of the image of f and a basis of its kernel.
  (c) Find a system of linearly independent equations of the kernel and the image of f.

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(3) Given the following system of equations,

$$\left.\begin{aligned} x + y + z &= 1 \\ 2x + ay + 2z &= b \\ ax + y + z &= 1 \end{aligned}\right\}$$

  (a) Discuss the system according to the different values of a and b.
  (b) Solve the system for the case $a = 2$, $b = 2$.

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(4) Consider the matrix

$$A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 3 \end{pmatrix}$$

  (a) Find the characteristic polynomial and the eigenvalues.
  (b) Show that the matrix is diagonalizable.
  (c) Find the corresponding diagonal form and the matrix change of basis.

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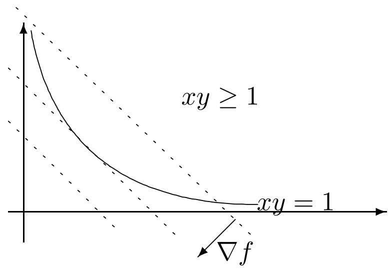
(5) Given the set $\mathcal{D} \subset \mathbb{R}^2$ defined by

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, xy \geq 1\}$$

  (a) Draw the set $\mathcal{D}$.
  (b) Draw the level curves of the function

$$f(x, y) = \frac{1}{x + y}$$

  (c) Using parts (a) and (b), argue whether the function f attains a maximum or a minimum value on $\mathcal{D}$. (Note that you are not asked to compute these values)



(a) En el gráfico el conjunto $\mathcal{D}$ es el que esta por encima de la curva de nivel $xy = 1$. Observemos que es la parte por encima del gráfico de la función $y = 1/x$ con $x > 0$.

(b) Las curvas de nivel de la función f están representadas por las líneas rectas discontinuas. Obsérvese que la curva de nivel dada for $f(x, y) = c$ coincide con la curva de nivel de $x + y = 1/c$, que representa a la recta $y = 1/c - x$.

(c) En la figura se ha representado el gradiente ∇f, que como se sabe apunta en la dirección en la que la función crece. Como se ve la función alcanza un máximo en el punto en el que la curva de nivel de f es tangente a la curva $xy = 1$, ya que se observa gráficamente que en todas las demás curvas de nivel de f que intersectan a $\mathcal{D}$, el valor de f es más pequeño.

Por otra parte, partiendo de cualquier punto de $\mathcal{D}$ y moviéndonos en la dirección de $-\nabla f$ permanecemos dentro del conjunto $\mathcal{D}$ y la función f decrece estrictamente, por lo que no se alcanza ningún mínimo en $\mathcal{D}$.

(6) Consider the function $f(x, y) = -8ax^2 - 2by^2 + cxy + 5x - 3y + 2$.
   (a) Discuss, according to the values of the parameters a, b and c, when is f strictly concave, knowing that $ab = 1$, $a \geq 0$, $b \geq 0$.
   (b) Compute the values of a, b and c assuming that the Taylor polynomial of f of second order about the point $(1, 0)$ is

$$-x^2 - 16y^2 + 5x - 3y + 2$$

(7) Given the function $f(x, y) = -5x^2 - 8y^2 - 2xy + 42x + 102y$.
   (a) Find the critical points and classify them.
   (b) Find the largest open set in which f is strictly concave and strictly convex.
   (c) Find the absolute maxima and minima.

(8) Consider the function $f(x, y, z) = x^2 - y^2 + 11z^2$ and the set $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 9\}$.
   (a) Write the Lagrange equations satisfied by the extreme points of f on the set M.
   (b) Determine the extreme points of f on el set M.