University Carlos III
Department of Economics
Mathematics II. Final Exam. June 2008

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.
- Read the exam carefully. Each part of the exam counts 1 point. Please, check that there are 10 pages in this booklet

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Consider the following system of linear equations,

$$\begin{cases} x + 2y + (a-1)\,z &= 1 \\ x + ay + z &= 1 \\ (a-1)\,x + 2y + z &= -3 \end{cases}$$

where $a \in \mathbb{R}$ is a parameter.

(a) Classify the above system according to the values of the parameter a.

(b) Solve the above system for the values of a for which the system is underdetermined. How many parameters are needed to describe the solution?

(2) Consider the matrix

$$A = \begin{pmatrix} 0 & -4 & 3 \\ -4 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}.$$

(a) Find the characteristic polynomial and the eigenvalues of the matrix A.

(b) Justify whether the matrix A is diagonalizable. And, if so, find two matrices D and P such that $A = PDP^{-1}$.

(c) Show how you can compute A^{200}. (It is enough to write it as the product of three matrices).

(3) Given the linear map $f : \mathbb{R}^4 \to \mathbb{R}^4$,

$$f(x, y, z, t) = (x + z, 2x + y + 2z + 2t, y + 2t, 3x + 2y + 3z + 4t)$$

(a) Write down the matrix of f (with respect to the canonical basis of $\mathbb{R}^4$). Compute the dimensions of the kernel and the image of f.

(b) Describe a homogeneous system of equations that determines the kernel of f and a homogeneous system of equations that determines the image of f. What is the minimum number of equations necessary to describe each of these systems?

(c) Find a basis of the image of f and a basis of the kernel of f.

(4) Consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \ln x, 1 \leq x \leq 2\}.$$

(a) Draw the set A, its boundary and its interior. Discuss whether the set A is open, closed, bounded, compact and/or convex. You must explain your answer.
(b) Prove that the function $f(x, y) = y^2 + (x - 1)^2$ has a maximum and a minimum on A.
(c) Using the level curves of $f(\cdot)$, find the maximum and the minimum of f on A.

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(5) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = \begin{cases} \frac{2x^2}{|x|+3|y|} & \text{si } (x, y) \neq (0, 0), \\ 0 & \text{si } (x, y) = (0, 0). \end{cases}$$

(a) Study if the function f is continuous at the point $(0, 0)$. Study at which points of $\mathbb{R}^2$ the function f is continuous.

(b) Compute the partial derivatives of f at the point $(0, 0)$, if they exist.

(6) Given the quadratic form

$$Q(x,y,z) = x^2 + 2ay^2 + z^2 + 2axy + 4ayz$$

(a) Determine the matrix associated to the above quadratic form.
(b) Classify the above quadratic form, according to the values of the parameter a.

(7) Consider the function

$$f(x, y) = x^3 + y^3 - 3xy.$$

(a) Determine the critical points of f and classify them.

(b) Determine the convex and open sets in $\mathbb{R}^2$ where the function f is concave, and the convex and open sets in $\mathbb{R}^2$ where the function f is convex.

(c) Study if f attains any global extreme points on the set $A = \{(x, y) \in \mathbb{R}^2 | x > \frac{2}{3}, y > \frac{2}{3}\}$.

(8) Consider the function

$$f(x, y) = xe^x$$

and the set

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}.$$

(a) Find the Lagrange equations that determine the extreme points of f on the set A.

(b) Determine the points that satisfy the Lagrange equations and find the extreme points of f on A, if they exist. Specify whether the extremum points correspond to a global maximum or minimum. (Hint: $2e^{-2} < e^{-1}$)
