University Carlos III of Madrid

Department of Economics
Mathematics II. Final Exam. June 2007.

IMPORTANT:

- **DURATION OF THE EXAM: 2h. 30min.**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam. These last two pages will not be graded. **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.
- Each part of the exam counts $0'5$ points.

Last Name: **Name:**

DNI: **Group:**

(1) Consider the following system of linear equations,

$$\begin{aligned} x + 3y + 2z &= 1 \\ 3x + y + 2z &= b \\ x + y + az &= 2b \end{aligned}$$

where $a, b \in \mathbb{R}$ are parameters.
 (a) Classify the system according to the values of the parameters a, b.
 (b) Solve the above system for the values of $a = 1$ and $b = 1/7$.

(2) Given the matrix

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

 (a) Find the characteristic polynomial and the eigenvalues.
 (b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.
 (c) Compute A^{10}. (You may use that $2^{10} = 1024$.)

(3) Given the linear map $f : \mathbb{R}^3 \to \mathbb{R}^4$,

$$f(x, y, z) = (2x - y + z, x - y, 3x - 2y + z, y + z)$$

 (a) Compute the matrix of f with respect to the canonical bases.
 (b) Compute the dimensions of the kernel and the image and a set of equations for these subspaces.
 (c) Find a basis of the image of f and a basis of the kernel of f.

(4) Given the set

$$A = \{(x, y) \in \mathbb{R}^2 : \quad x - 2y \geq -6, \quad x \leq 0, \quad y \geq 0\}$$

 (a) Draw the set A, computing its intersection with the axes. Draw its boundary and interior and discuss whether the set A is open, closed, bounded, compact and/or convex. You must explain your answer.

(b) Show that the function

$$f(x,y) = \frac{x^2}{(x+4)^2 + (y-2)^2}$$

attains a maximum and a minimum on the set A.

(c) Draw the level curves of the function $g(x,y) = y + x^2$ and use them to determine the maxima and minima of g on A.

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(5) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{x^2+x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Study if the function f is continuous at the point $(0,0)$. Study at which points of $\mathbb{R}^2$ the function f is continuous.

(b) Compute the partial derivatives of f at the point $(0,0)$.

(c) At which points of $\mathbb{R}^2$ is the function f differentiable?

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(6) Consider the function

$$f(x,y) = 2x^4 + y^4 - 2x^2 - 2y^2$$

(a) Determine the critical points of f.

(b) Classify the critical points of f that you obtained in part (a).

(c) Determine the largest open set A of $\mathbb{R}^2$ where the function f is concave.

(d) Find the global extreme points of f on A.

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(7) Consider the function

$$f(x,y,z) = 2x^2 + y^2 - x - z + 4z^2$$

and the set

$$A = \{(x,y,z) : x + y = z\}$$

(a) Find the Lagrange equations that determine the extreme points of f in the set A.

(b) Determine the points that satisfy the Lagrange equations and find the extreme points of f, specifying whether they correspond to a maximum or minimum.
