University Carlos III of Madrid

Department of Economics
Mathematics II. Final Exam. June 2006.

IMPORTANT:

- **DURATION OF THE EXAM: 2h. 30min.**
- Calculators are **NOT** allowed.
- **Hand in this booklet.** Do not hand in scrap paper. Only the answers written on this booklet will be graded.
- You must show a valid ID to the professor.
- Each part of the exam counts $0'5$ points.

Last Name: **Name:**

DNI: **Group:**

(1) Consider the following system of linear equations,

$$\begin{aligned} ax + y + z &= 1 \\ x + ay + z &= 0 \\ x + y + az &= -1 \end{aligned}$$

where $a \in \mathbb{R}$ is a parameter.

(a) Classify the system according to the values of a.
(b) Solve the above system for the values of a for which the system is compatible indeterminate.

(2) Given the matrix

$$A = \begin{pmatrix} 3 & 0 & \beta \\ 2 & -1 & -4 \\ 0 & 0 & \alpha \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{R}$ are parameters,

(a) Find the characteristic polynomial and the eigenvalues.
(b) Determine for which values of the parameters $\alpha, \beta \in \mathbb{R}$, the matrix is diagonalizable.
(c) For the values of the parameters $\alpha = -1$ and $\beta = -8$, find the corresponding diagonal matrix and the matrix change of basis.

(3) Given the linear mapping $f : \mathbb{R}^4 \to \mathbb{R}^3$,

$$f(x, y, z, t) = (-x - y - z, x - y - t, -3x - 3y - 3z)$$

(a) Compute the dimensions of the kernel and the image and a set of equations for these subspaces.
(b) Find a basis of the image of f and a basis of the kernel of f.

(4) Given the set

$$S = \{(x, y) \in \mathbb{R}^2 : x \leq 1, y \geq 0, y \leq x^3\}$$

(a) Draw the set S, its boundary and interior and discuss whether the set S is open, closed, bounded, compact and/or convex. You must explain your answer.
(b) Show that the function $f(x, y) = (x - 1)^2 + (y - 1)^2$ has a maximum and a minimum on the set S.
(c) Draw the level curves of $f(x, y)$ and determine where the maxima and the minima of f on S.

(5) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = \begin{cases} \frac{3x^2y^2}{x^4+y^4} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Study if the function f is continuous at the point $(0, 0)$.
(b) Compute the partial derivatives of f at the point $(0, 0)$.
(c) Determine at which points of $\mathbb{R}^2$ the partial derivatives of f are continuous.

(6) Consider the function

$$f(x, y, z) = x^2 + ay^2 + z^2 + 2axy + 2xz - 2yz$$

(a) Find the Hessian matrix of f.

(b) Study for which values of parameter a, the function f is strictly concave or strictly convex.

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(7) Consider the function $f(x,y) = 8x^3 + 2xy - 3x^2 + y^2 + 1$.
(a) Find the critical points of f.
(b) Classify the critical points of f that you found in the previous part.
(c) Determine whether f has any global extreme points on the set

$$A = \{(x,y) \in \mathbb{R}^2 : \frac{1}{4} < x\}$$

Hint: Study the concavity or convexity of f on the set A.

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(8) Consider the function

$$f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} + 2x^2$$

and the set

$$A = \{(x,y) : x^2 + y^2 = 3\}$$

(a) Find the Lagrange equations that determine the extreme points of f in the set A.
(b) Determine the points that satisfy the Lagrange equations and find the extreme points of f in A, specifying whether they are maxima or minima.
