University Carlos III of Madrid

Department of Economics
Mathematics II. Final Exam. June 2003.

Last Name: **Name:**

DNI: **Group:**

(1) Consider the linear system of equations, **1 point**

$$\left.\begin{array}{r} x + az = b \\ ax + 2y + z = 1 \\ x + y + az = 1 \end{array}\right\}$$

(a) Discuss for what values of a and b the system is compatible (determinate and/or indeterminate).
(b) Solve the system for the case $a = 1$, $b = 1$.

(2) Given the linear mapping $f : \mathbb{R}^3 \to \mathbb{R}^4$ **1'5 points**

$$f(x, y, z) = (x + y + z, x + y - z, x + y + 3z, z)$$

(a) Find the matrix of f with respect the canonical bases and find the dimension of its image and its kernel.
(b) Find a basis of the image of f and a basis of its kernel.
(c) Find a set of independent linear equations of the image and the kernel of f.

(3) Given the following matrix **1'5 points**

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(a) Find the characteristic polynomial and the eigenvalues.

(b) Assuming the matrix is diagonalizable, find its diagonal form and the matrix change of basis.

(c) Compute the n-th power A^n of A. (Leave the answer indicated as the product of three matrices, without computing the inverse matrix)

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(4) Consider the set $\mathcal{D} \subset \mathbb{R}^2$ defined by **1'5 points**

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \mid y \geq x^4, y \leq 2x^3\}$$

(a) Draw the set $\mathcal{D}$ and find the intersection points of the curves that define it.
(b) Study whether $\mathcal{D}$ is *closed, open, bounded, compact and convex*. Justify the answer.
(c) Find the values of b for which you can assure that the function

$$f(x, y) = \frac{1}{(x-5)^2} + \frac{1}{(y-b)^2}$$

attains a maximum and a minimum value on $\mathcal{D}$. Justify the answer.

(5) Let $f(x,y) = x^{\alpha}y^{1-\alpha}$, with $0 < \alpha < 1$ be the Cobb-Douglas production function. $\boxed{\textbf{1 point}}$

(a) Assuming that the direction of maximal growth of f at the point $(1,1)$ is determined by the unit vector $u = \frac{1}{\sqrt{10}}(1,3)$, determine α.

(b) Assuming that the tangent line to the level curve of f that goes through the point $(1,1)$ is $x + 2y = 3$, find α.

Note: The values of α in parts (a) and (b) may not coincide.

(6) Consider the function $f(x, y) = 4ax^2 - 2by^2 - xy + 3y + 4x + 1$. **1 point**

(a) Discuss, accordingly to the values of the parameters a and b, when is f strictly concave.

(b) Compute the second order Taylor polynomial of the function $g(x, y) = x^3 + 4x^2 + 2y^2 - xy + 3y + 4x + 1$ about the point $(0, 0)$.

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(7) Consider the function $f(x, y) = ax^2 + y^3 - 2x - 3y$. **1 point**

(a) Find the critical points of f and classify them depending on the values of a, for $a \neq 0$.

(b) Consider the set $A = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. Discuss according to the values of a if f attains an extreme value on A.

(8) Let **1'5 points**

$$f(x, y) = 4x - 2x^2 - 2y^2$$

and

$$S = \{(x, y) : x^2 + y^2 \leq 25\}.$$

(a) Find the critical points of f in the interior of S.
(b) Find the points on the boundary of S, that satisfy the Lagrange equations for f.
(c) Find the global extreme points of f on the set S.
