UNIVERSIDAD CARLOS III DE MADRID
MATHEMATICS II
EXERCISES

CHAPTER 5: Optimization

5-1. Find and classify the critical points of the following functions.
(a) $f(x,y)=x^2-y^2+xy$.
(b) $f(x,y)=x^2+y^2+2xy$.
(c) $f(x,y)=e^{x\cos y}$.
(d) $f(x,y)=e^{1+x^2-y^2}$.
(e) $f(x,y)=x\sin y$.
(f) $f(x,y)=xe^{-x}(y^2-4y)$

5-2. Find the critical points of the following functions. For which points the second derivative criterion does not give any information?
(a) $f(x,y)=x^3+y^3$.
(b) $f(x,y)=\left((x-1)^2+(y+2)^2\right)^{1/2}$.
(c) $f(x,y)=x^3+y^3-3x^2+6y^2+3x+12y+7$.
(d) $f(x,y)=x^{2/3}+y^{2/3}$

5-3. Let $f(x,y)=(3-x)(3-y)(x+y-3)$.
(a) Find and classify the critical points.
(b) ¿Does f have absolute extrema? (hint: consider the line $y=x$)

5-4. Find the values of the constants a, b and c so that the function $f(x,y)=ax^2y+bxy+2xy^2+c$ has a local minimum at the point $(2/3,1/3)$ and the minimum value at that point is $-1/9$.

5-5. The income function is $R(x,y)=x(100-6x)+y(192-4y)$ where x and y are the number of articles sold. If the cost function is $C(x,y)=2x^2+2y^2+4xy-8x+20$ determine the maximum profit.

5-6. A milk store produces x units of whole milk and y units of skim milk. The price for whole milk is $p(x)=100-x$ and the price for skim milk is $q(y)=100-y$. The cost of production is $C(x,y)=x^2+xy+y^2$. How should the company choose x and y to maximize profits?

5-7. A monopolist produces a good which is bought by two types of consumers. The consumers of type 1 are willing to pay $50-5q_1$ euros in order to purchase q_1 units of the good. The consumers of type 2 are willing to pay $100-10q_2$ euros in order to purchase q_2 units of the good. The cost function of the monopolist is $c(q)=90+20q$ euros. How much should the monopolist produce in each market?

5-8. Find and classify the extreme points of the following functions under the given restrictions.
(a) $f(x,,y,z)=x+y+z$ in $x^2+y^2+z^2=2$.
(b) $f(x,,y)=\cos(x^2-y^2)$ in $x^2+y^2=1$.

5-9. Minimize $x^4+y^4+z^4$ on the plane $x+y+z=1$.

5-10. A company makes two products, P_1 and P_2. If the company sells x_1 units of P_1 and x_2 units of P_2 it receives a net profit of $R=-5x_1^2-8x_2^2-2x_1x_2+42x_1+102x_2$. Find x_1 and x_2 that maximize net profit.

5-11. The prices for two goods produced by a monopolist are

$$p_1 = 256-3q_1-q_2$$
$$p_2 = 222+q_1-5q_2$$

where p_1,p_2 are the prices and q_1,q_2 are the quantities produced. The cost function is $C(q_1,q_2)=q_1^2+q_1q_2+q_2^2$. Find the quantities that maximize profit.

5-12. The production function for a firm is $4x+xy+2yz$, where x is labor and y is capital. The total budget that the company can spend is 2000\$. Each unit of labor costs 20\$, whereas each unit of capital costs 4\$. Find the optimal level of production for the firm.

5-13. An editor has been assigned a budget of 60.000 to be spent on advertising and production of a new book. She estimates that spending x thousand euro in production and y thousand euro in advertising she can sell $f(x, y) = 20x^{3/2}y$ books. If she wants to maximize sales, how much should she allocate to advertising and how much should she allocate to production?

5-14. A store sells two products which are close substitutes. The manager has found that if he sells the products at the prices P_1 and P_2, the net profit is $R = 500P_1 + 800P_2 + 1,5P_1P_2 - 1,5P_1^2 - P_2^2$. Find the optimal prices for the manager.

5-15. The utility function of a consumer is $u(x, y) = \frac{1}{3}\ln x + \frac{2}{3}\ln y$, where x and y are the consumption goods, with prices, respectively, p_1 and p_2. The rent of the agent is M. Find the demand of the agent for each good.

5-16. Find and classify the extreme points of the function f on the given set.
(a) $f(x, y, z) = x^2 + y^2 + z^2$ on the set $\{(x, y, z) \in \mathbb{R}^3 : x + 2y + z = 1, 2x - 3y - z = 4\}$.
(b) $f(x, y, z) = (y + z - 3)^2$ on the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y + z = 2, x + y^2 + 2z = 2\}$.
(c) $f(x, y, z) = x + y + z$ on the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, x - y - z = 1\}$.
(d) $f(x, y, z) = x^2 + y^2 + z^2$ on the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z, x + y + z = 4\}$.

5-17. Find the maximum of the function $f(x, y, z) = xyz$ on the set $\{(x, y, z) \in \mathbb{R}^3 : x + y + z \leq 1, \quad x, y, z \geq 0\}$.

5-18. Find the minimum of the function $f(x, y) = 2y - x^2$ on the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, \quad x, y \geq 0\}$.

5-19. Solve the optimization problem

$$\begin{cases} \text{min} & x^2 + y^2 - 20x \\ \text{s.t.} & 25x^2 + 4y^2 \leq 100 \end{cases}$$

5-20. Solve the optimization problem

$$\begin{cases} \text{max} & x + y - 2z \\ \text{s.t.} & z \geq x^2 + y^2 \\ & x, y, z \geq 0 \end{cases}$$