CHAPTER 5: Optimization

5-1. *Find and classify the critical points of the following functions.*
(a) $f(x,y) = x^2 - y^2 + xy$.
(b) $f(x,y) = x^2 + y^2 + 2xy$.
(c) $f(x,y) = e^{x\cos y}$.
(d) $f(x,y) = e^{1+x^2-y^2}$.
(e) $f(x,y) = x\sin y$.
(f) $f(x,y) = xe^{-x}(y^2 - 4y)$

Solution:
(a) The gradient of $f(x,y) = x^2 - y^2 + xy$ is

$$\nabla f(x,y) = (2x + y, -2y + x)$$

and the critical points are the solutions to the following system of equations

$$\begin{aligned} 2x + y &= 0 \\ -2y + x &= 0 \end{aligned}$$

whose solution is $(0,0)$. The Hessian is

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

which is indefinite. Hence, $(0,0)$ is a saddle point.
(b) The gradient of $f(x,y) = x^2 + y^2 + 2xy$ is $\nabla f(x,y) = (2x + 2y, 2y + 2x)$. The critical points are the points of the form $y = -x$. The Hessian is

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

which is positive semidefinite. The second order conditions are not informative. But, since $f(x,y) = (x+y)^2 \geq 0$ and $f(x,-x) = 0$ we see that the points of the form $(x,-x)$ are global minima of f .
(c) The gradient of $f(x,y) = e^{x\cos y}$ is

$$\nabla f(x,y) = (e^{x\cos y}\cos y, -xe^{x\cos y}\sin y)$$

The critical points are solutions to the following system of equations

$$\left.\begin{aligned} (\cos y)\, e^{x\cos y} &= 0 \\ -x\,(\sin y)\, e^{x\cos y} &= 0 \end{aligned}\right\} \qquad \equiv \qquad \left.\begin{aligned} \cos y &= 0 \\ -x\sin y &= 0 \end{aligned}\right\}$$

The first equation implies that

$$y = \frac{\pi}{2} + k\pi, \qquad k = 0, \pm 1, \pm 2, \ldots$$

For those values of y, $\sin y \neq 0$ and the second equation implies now that $x = 0$. The solutions are

$$(0, \frac{\pi}{2} + k\pi) \qquad k = 0, \pm 1, \pm 2, \ldots$$

The Hessian is

$$e^{x\cos y}\begin{pmatrix} \cos^2 y & -\sin y - x\sin y\cos y \\ -\sin y - x\sin y\cos y & -x\cos y + x^2\sin^2 y \end{pmatrix}$$

For $x = 0$, $y = \frac{\pi}{2} + k\pi$, we obtain that

$$\left.\begin{pmatrix} 0 & -\sin y \\ -\sin y & 0 \end{pmatrix}\right|_{y=\frac{\pi}{2}+k\pi}$$

whose determinant is $-\sin^2(\frac{\pi}{2} + k\pi) = -1$. Thus, the critical points are saddle points.

(d) The gradient of $f(x,y) = e^{1+x^2-y^2}$ is

$$\nabla f(x,y) = 2e^{1+x^2-y^2}(x,-y)$$

and the unique critical point is $(0,0)$. The Hessian is

$$\mathrm{H}f(0,0) = e^{1+x^2-y^2}\begin{pmatrix} 2+4x^2 & -4yx \\ -4yx & -2+4y^2 \end{pmatrix}\Bigg|_{x=0,y=0} = \begin{pmatrix} 2e & 0 \\ 0 & -2e \end{pmatrix}$$

which is indefinite, so $(0,0)$ is a saddle point.

(e) The gradient of $f(x,y) = x\sin y$ is

$$\nabla f(x,y) = (\sin y, x\cos y)$$

and the critical points are solutions to the following system of equations

$$\begin{aligned} \sin y &= 0 \\ x\cos y &= 0 \end{aligned}$$

From the first equation we see that

$$y = k\pi, \qquad k = 0, \pm 1, \pm 2, \dots$$

so $\cos y = \pm 1$ and the second equation implies that $x = 0$. The solutions are

$$(0, k\pi) \qquad k = 0, \pm 1, \pm 2, \dots$$

The Hessian is

$$\begin{pmatrix} 0 & \cos y \\ \cos y & -x\sin y \end{pmatrix}\Bigg|_{x=0,y=k\pi} = \begin{pmatrix} 0 & \cos(k\pi) \\ \cos(k\pi) & 0 \end{pmatrix}$$

whose determinant is $-\cos^2(k\pi) = 1$. Hence, the Hessian is indefinite and the critical points are saddle points.

(f) The gradient of $f(x,y) = xe^{-x}(y^2-4y)$ is

$$\nabla f(x,y) = e^{-x}((1-x)(y^2-4y), x(2y-4))$$

The critical points are solutions to the following system of equations

$$\begin{aligned} (1-x)(y^2-4y) &= 0 \\ x(2y-4) &= 0 \end{aligned}$$

that is $(0,0)$, $(0,4)$ and $(1,2)$. The Hessian is

$$\mathrm{H}f(x,y) = e^{-x}\begin{pmatrix} (x-2)(y^2-4y) & (1-x)(2y-4) \\ (1-x)(2y-4) & 2x \end{pmatrix}$$

At the critical points we obtain that

$$\begin{aligned} \mathrm{H}f(0,0) &= \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \text{(indefinite)} \\ \mathrm{H}f(0,4) &= \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \text{(indefinite)} \\ \mathrm{H}f(1,2) &= \begin{pmatrix} 4e^{-1} & 0 \\ 0 & 2e^{-1} \end{pmatrix} \text{(positive definite)} \end{aligned}$$

so $(0,0)$, $(0,4)$ are saddle points and $(1,2)$ is a local minimum.

5-2. *Find the critical points of the following functions. For which points the second derivative criterion does not give any information?*

(a) $f(x,y) = x^3 + y^3$.

(b) $f(x,y) = \left((x-1)^2 + (y+2)^2\right)^{1/2}$.

(c) $f(x,y) = x^3 + y^3 - 3x^2 + 6y^2 + 3x + 12y + 7$.

(d) $f(x,y) = x^{2/3} + y^{2/3}$

Solution:

(a) The gradient of $f(x,y)=x^3+y^3$ is $\nabla f(x,y)=3(x^2,y^2)$. The unique critical point is $(0,0)$. The Hessian is

$$\mathrm{H}f(0,0)=\left(\begin{array}{cc} 6x & 0 \\ 0 & 6y \end{array}\right)\Big|_{x=0,y=0}=\left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)$$

The second derivative criterion does not provide any information. But, noting that $f(0,0)=0$ and that

$$f(x,0)=x^3\begin{cases} >0 & \text{if } x>0, \\ <0 & \text{if } x<0. \end{cases}$$

we see that $(0,0)$ is a saddle point.

(b) The gradient of $f(x,y)=((x-1)^2+(y+2)^2)^{1/2}$ is

$$\nabla f(x,y)=\frac{1}{\sqrt{(x-1)^2+(y+2)^2}}(x-1,y+2)$$

which does not vanish in its domain. Note that f is differentiable, except at the point $(1,-2)$. Therefore, the unique critical point is $(1,-2)$ and since the function is not differentiable at this point we may not apply the second derivative criterion.
Noting that $f(1,-2)=0<((x-1)^2+(y+2)^2)^{1/2}$ if $(x,y)\neq(1,-2)$, we see that $(1,-2)$ is a global minimum.

(c) The gradient of $f(x,y)=x^3+y^3-3x^2+6y^2+3x+12y+7$ is

$$\nabla f(x,y)=(3x^2-6x+3,3y^2+12y+12)$$

The critical points are solutions of the system

$$x^2-2x+1=0$$
$$y^2+4y+4=0$$

whose unique solution is $x=1$, $y=-2$. The unique critical point is $(1,-2)$. The Hessian is

$$\mathrm{H}f(1,-2)=\left(\begin{array}{cc} 6x-6 & 0 \\ 0 & 6y+12 \end{array}\right)\Big|_{x=1,y=-2}=\left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)$$

The second derivative criterion does not provide any information. Note that f is a polynomial of degree 3, $\nabla f(1,-2)=(0,0$ and also $\mathrm{H}f(1,-2)=0$. Thus, we see that $f(x,y)=(y+2)^3+(x-1)^3$. And we have that

$$f(x+1,-2)=x^3\begin{cases} >0 & \text{if } x>0, \\ <0 & \text{if } x<0. \end{cases}$$

So, $(1,-2)$ is a saddle point.

(d) The function $f(x,y)=x^{2/3}+y^{2/3}$ is not differentiable at $(0,0)$. At the points $(x,y)\neq(0,0)$ we have that

$$\nabla f(x,y)=\left(\frac{2}{3\sqrt[3]{x}},\frac{2}{3\sqrt[3]{y}}\right)$$

so the function never vanishes in its domain. The unique critical point is $(0,0)$. The second derivative criterion does not provide any information. But, noting that $f(0,0)=0$ and that if $(x,y)\neq(0,0)$,

$$f(x,y)=x^{2/3}+y^{2/3}>0$$

we see that $(0,0)$ is a global minimum.

5-3. *Let* $f(x,y)=(3-x)(3-y)(x+y-3)$.
(a) *Find and classify the critical points.*
(b) *¿Does f have absolute extrema? (hint: consider the line $y=x$)*

Solution:

(a) The function $f(x,y)=(3-x)(3-y)(x+y-3)$ is differentiable in $\mathbb{R}^2$. The gradient is

$$\nabla f(x,y)=(-6x-9y+18+2xy+y^2,-9x-6y+18+x^2+2xy)$$

and the critical points are solutions of the system of equations

$$-6x-9y+18+2xy+y^2=0$$
$$-9x-6y+18+x^2+2xy=0$$

whose solutions are $(3,0)$, $(3,3)$, $(0,3)$ y $(2,2)$. The Hessian is

$$\mathrm{H}f(x,y)=\left(\begin{array}{cc} -6+2y & 2x+2y-9 \\ 2x+2y-9 & -6+2x \end{array}\right)$$

Which evaluated at the critical points yields

$$\mathrm{H}f(3,0) = \begin{pmatrix} -6 & -3 \\ -3 & 0 \end{pmatrix} \quad \text{which is indefinite}$$

$$\mathrm{H}f(3,3) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \quad \text{which is indefinite}$$

$$\mathrm{H}f(0,3) = \begin{pmatrix} 0 & -3 \\ -3 & -6 \end{pmatrix} \quad \text{which is indefinite}$$

$$\mathrm{H}f(2,2) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \quad \text{which is negative definite}$$

We obtain that $(2,2)$ is a maximum and $(3,0)$, $(0,3)$ y $(3,3)$ are saddle points.

(b) Using the suggestion $y = x$, we obtain that $f(x,x) = (3-x)^2(2x-3)$. Note that

$$\lim_{x\to+\infty} f(x,x) = +\infty \qquad \lim_{x\to-\infty} f(x,x) = -\infty$$

so f does not attain neither a maximum nor a minimum in $\mathbb{R}^2$.

5-4. *Find the values of the constants a, b and c so that the function $f(x,y) = ax^2y + bxy + 2xy^2 + c$ has a local minimum at the point $(2/3, 1/3)$ and the minimum value at that point is $-1/9$.*

Solution: The gradient of $f(x,y) = ax^2y + bxy + 2xy^2 + c$ is

$$\nabla f(x,y) = (2axy + by + 2y^2, ax^2 + bx + 4xy)$$

The critical points are solution of the system of equations

$$\begin{aligned} 2axy + by + 2y^2 &= 0 \\ ax^2 + bx + 4xy &= 0 \end{aligned}$$

Therefore, the critical points are

$$(0,0), (-\frac{b}{a}, 0), (0, -\frac{b}{2}), (-\frac{b}{3a}, -\frac{b}{6})$$

The point $(2/3, 1/3)$ is a critical point if

$$(2/3, 1/3) = (-\frac{b}{3a}, -\frac{b}{6})$$

This happens if $a = 1$, $b = -2$.

Now, we choose c such that the value of the function at the point $(2/3, 1/3)$ is $-1/9$.

$$f(2/3, 1/3) = \left(x^2y - 2xy + 2xy^2 + c\right)\Big|_{x=2/3, y=1/3} = -\frac{4}{27} + c = -1/9$$

We obtain $c = \frac{1}{27}$.

We check that $(2/3, 1/3)$ is a minimum. The Hessian is $\mathrm{H}f(2/3,1/3) = \begin{pmatrix} 2y & 2x - 2 + 4y \\ 2x - 2 + 4y & 4x \end{pmatrix}\Big|_{x=2/3, y=1/3} =$ $\begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{8}{3} \end{pmatrix}$ which is positive definite. Therefore $(2/3, 1/3)$ is a minimum.

5-5. *The income function is $R(x,y) = x(100 - 6x) + y(192 - 4y)$ where x and y are the number of articles sold. If the cost function is $C(x,y) = 2x^2 + 2y^2 + 4xy - 8x + 20$ determine the maximum profit.*

Solution: The profit is

$$B(x,y) = R(x,y) - C(x,y) = x(100 - 6x) + y(192 - 4y) - (2x^2 + 2y^2 + 4xy - 8x + 20)$$

whose gradient is

$$\nabla B(x,y) = (108 - 16x - 4y, 192 - 4x - 12y)$$

We obtain the critical point $(3, 15)$. The Hessian

$$\mathrm{H}B(x,y) = \begin{pmatrix} -16 & -4 \\ -4 & -12 \end{pmatrix}$$

is negative definite in $\mathbb{R}^2$. The function is concave in $\mathbb{R}^2$ and hence $(3, 15)$ is a global maximum.

5-6. *A milk store produces x units of whole milk and y units of skim milk. The price for whole milk is $p(x) = 100-x$ and the price for skim milk is $q(y) = 100 - y$. The cost of production is $C(x, y) = x^2 + xy + y^2$. How should the company choose x and y to maximize profits?*

Solution: The profit is

$$B(x, y) = x(100 - x) + y(100 - y) - (x^2 + xy + y^2)$$

The unique critical point is $(20, 20)$. Since, the Hessian is $\begin{pmatrix} -4 & -1 \\ -1 & -4 \end{pmatrix}$ which is negative definite, the critical point is a maximum. Since in addition the function is concave, the critical point is a global maximum.

5-7. *A monopolist produces a good which is bought by two types of consumers. The consumers of type 1 are willing to pay $50 - 5q_1$ euros in order to purchase q_1 units of the good. The consumers of type 2 are willing to pay $100 - 10q_2$ euros in order to purchase q_2 units of the good. The cost function of the monopolist is $c(q) = 90 + 20q$ euros. How much should the monopolist produce in each market?*

Solution: If the monopolist sells q_1 units in the first market and q_2 units in the second market, the profit is

$$q_1(50 - 5q_1) + q_2(100 - 10q_2) - 90 - 20(q_1 + q_2) = 30q_1 - 5q_1^2 + 80q_2 - 10q_2^2 - 90$$

The first order conditions are

$$\begin{aligned} 30 &= 10q_1 \\ 80 &= 20q_2 \end{aligned}$$

so the solution is

$$q_1 = 3, \quad q_2 = 4$$

5-8. *Find and classify the extreme points of the following functions under the given restrictions.*
(a) *$f(x, y, z) = x + y + z$ in $x^2 + y^2 + z^2 = 2$.*
(b) *$f(x, y) = \cos(x^2 - y^2)$ in $x^2 + y^2 = 1$.*

Solution:
(a) The Lagrangian is

$$L(x, y, z) = f(x, y, z) - \lambda g(x, y, z)$$

We obtain the Lagrange equations

$$\begin{aligned} D_x(L(x, y, z)) &= 1 - 2\lambda x = 0 \\ D_y(L(x, y, z)) &= 1 - 2\lambda y = 0 \\ D_z(L(x, y, z)) &= 1 - 2\lambda z = 0 \\ x^2 + y^2 + z^2 &= 2 \end{aligned}$$

Comparing the first three equations we see that $x = y = z$. Thus, $3x^2 = 2$, that is

$$x = \pm\sqrt{\frac{2}{3}}$$

(b) The Lagrangian is

$$L(x, y) = f(x, y) - \lambda g(x, y)$$

The Lagrange equations may be written as

$$\begin{aligned} &x(\sin(-x^2 + y^2) - \lambda) = 0 \\ &y(\sin(-x^2 + y^2) + \lambda) = 0 \\ &x^2 + y^2 = 1 \end{aligned}$$

We see that $(0, \pm 1)$ y $(\pm 1, 0)$ are solutions. If we now, assume that $x \neq 0$, $y \neq 0$, from the first two equations we obtain that

$$\begin{aligned} \lambda &= \sin\left(-x^2 + y^2\right) \\ \lambda &= -\sin\left(-x^2 + y^2\right) \end{aligned}$$

Therefore,

$$\sin\left(-x^2 + y^2\right) = -\sin\left(-x^2 + y^2\right)$$

from where
$$x^2 = y^2 = \frac{1}{2}$$
and hence $x = y$ or $x = -y$. The solutions are
$$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$$
Substituting in the objective function we see that $(0, \pm 1)$ y $(\pm 1, 0)$ are minima and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, are maxima.

5-9. *Minimize $x^4 + y^4 + z^4$ on the plane $x + y + z = 1$.*

Solution: The Lagrangian is
$$L(x, y, z) = f(x, y, z) - \lambda g(x, y, z)$$
The Lagrange equations are
$$\begin{aligned} D_x(L(x, y, z)) =& 4x^3 - \lambda = 0 \\ D_y(L(x, y, z)) =& 4y^3 - \lambda = 0 \\ D_z(L(x, y, z)) =& 4z^3 - \lambda = 0 \\ & x + y + x - z = 1 \end{aligned}$$
and we obtain that $x = y = z$. Substituting in the last equation we obtain that $3x = 1$. Hence,
$$x = y = z = \frac{1}{3}$$
Note that the Hessian is $HL = \begin{pmatrix} 12x^2 & 0 & 0 \\ 0 & 12y^2 & 0 \\ 0 & 0 & 12z^2 \end{pmatrix}\Bigg|_{x=1/3, y=1/3, z=1/3} = \begin{pmatrix} \frac{4}{3} & 0 & 0 \\ 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$ which is positive definite and therefore this point is a local minimum.

5-10. *A company makes two products, P_1 and P_2. If the company sells x_1 units of P_1 and x_2 units of P_2 it receives a net profit of $R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$. Find x_1 and x_2 that maximize net profit.*

5-11. *The prices for two goods produced by a monopolist are*
$$\begin{aligned} p_1 =& 256 - 3q_1 - q_2 \\ p_2 =& 222 + q_1 - 5q_2 \end{aligned}$$
where p_1, p_2 are the prices and q_1, q_2 are the quantities produced. The cost function is $C(q_1, q_2) = q_1^2 + q_1q_2 + q_2^2$. Find the quantities that maximize profit.

5-12. *The production function for a firm is $4x + xy + 2y$, where x is labor and y is capital. The total budget that the company can spend is 2000\$. Each unit of labor costs 20\$, whereas each unit of capital costs 4\$. Find the optimal level of production for the firm.*

5-13. *An editor has been assigned a budget of 60.000 to be spent on advertising and production of a new book. She estimates that spending x thousand euro in production and y thousand euro in advertising she can sell $f(x, y) = 20x^{3/2}y$ books. If she wants to maximize sales, how much should she allocate to advertising and how much should she allocate to production?*

Solution: The editor's problem is
$$\left.\begin{array}{rc} \text{max.} & 20x^{3/2}y \\ \text{s.t.} & x + y = 60000 \\ x, y \geq 0 & \end{array}\right\}$$
The feasible set is compact and the objective function is continuous. Hence, there is a solution. The corner points $(0, 60000)$ and $(60000, 0)$ yield a value of 0 in the objective function. These points correspond to a minimum and we may assume the solution is interior
$$\left.\begin{array}{rc} \text{max.} & 20x^{3/2}y \\ \text{s.t.} & x + y = 60000 \end{array}\right\}$$

The Lagrangian is

$$L(x,y) = 20x^{3/2}y + \lambda(60000 - x - y)$$

We obtain the Lagrange equations

$$\begin{aligned} 30x^{1/2}y - \lambda &= 0 \\ 20x^{3/2} - \lambda &= 0 \\ x + y &= 60000 \end{aligned}$$

From the two first equations we obtain that

$$30x^{1/2}y = 20x^{3/2}$$

If we assume that $x \neq 0$, we obtain that $y = 2x/3$, so the solution is $x = 36000$, $y = 24000$. This point satisfies the constraints of the problem.

The Hessian matrix of L is

$$\mathrm{H}\, L(x,y) = \begin{pmatrix} \frac{15x}{\sqrt{x}} & 30\sqrt{x} \\ 30\sqrt{x} & 0 \end{pmatrix} = 15 \begin{pmatrix} \frac{y}{\sqrt{x}} & 2\sqrt{x} \\ 2\sqrt{x} & 0 \end{pmatrix}$$

At the point $(36000, 24000)$ we have that

$$\mathrm{H}\, L(36000, 24000) = 15 \begin{pmatrix} 40 & 120 \\ 120 & 0 \end{pmatrix} = 600 \begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix}$$

The restriction of the problem is $g(x,y) = x + y - 60000$ and $\nabla g(36000, 24000) = (1,1)$. Then,

$$\begin{aligned} T_{(36000,24000)}M &= \{(v_1,v_2) \in \mathbb{R}^2 : (v_1,v_2)\cdot(1,1) = 0\} \\ &= \{(v_1,v_2) \in \mathbb{R}^2 : v_1 + v_2 = 0\} \\ &= \{(t,-t) : t \in \mathbb{R}\} \end{aligned}$$

The quadratic form

$$(t,-t)\cdot \mathrm{H}\, L(36000,24000) \begin{pmatrix} t \\ -t \end{pmatrix} = 600 \left((t,-t)\cdot \begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} t \\ -t \end{pmatrix} \right) = -3000t^2$$

is negative definite. Hence, the point $(36000, 24000)$ is a local maximum. And by Weierstrass Theorem is also a global maximum.

5-14. *A store sells two products which are close substitutes. The manager has found that if he sells the products at the prices P_1 and P_2, the net profit is $R = 500P_1 + 800P_2 + 1,5P_1P_2 - 1,5P_1^2 - P_2^2$. Find the optimal prices for the manager.*

Solution: The store maximizes the function

$$R(P_1,P_2) = 500P_1 + 800P_2 + \frac{3}{2}P_1P_2 - \frac{3}{2}P_1^2 - P_2^2$$

The gradient is

$$\nabla R(P_1,P_2) = (500 + \frac{3}{2}P_2 - 3P_1, 800 + \frac{3}{2}P_1 - 2P_2)$$

The unique critical point is

$$P_1 = \frac{2800}{3}, \qquad P_2 = 600$$

since

$$\mathrm{H}R(P_1,P_2) = \begin{pmatrix} -3 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix}$$

satisfies $D_1 < 0$, $D_2 > 0$, it is negative definite. Therefore, the profit function is concave in $\mathbb{R}^2_{++}$ and the critical point is a global maximum.

5-15. *The utility function of a consumer is $u(x,y) = \frac{1}{3}\ln x + \frac{2}{3}\ln y$, where x and y are the consumption goods, with prices, respectively, p_1 and p_2. The rent of the agent is M. Find the demand of the agent for each good.*

Solution: The consumer demands the amounts of goods x and y which are solutions to the following problem,

$$\left.\begin{array}{ll} \text{max.} & \frac{1}{3}\ln x + \frac{2}{3}\ln y \\ \text{s.t.} & p_1x + p_2y = M \end{array}\right\}$$

The Lagrangian is

$$L(x,y) = \frac{1}{3}\ln x + \frac{2}{3}\ln y + \lambda(M - p_1x - p_2y)$$

We obtain the Lagrange equations

$$\begin{aligned}\frac{1}{3x}-\lambda p_1 =&0\\ \frac{2}{3y}-\lambda p_2 =&0\\ p_1x+p_2y =&M\end{aligned}$$

The first two equations may be written as

$$\begin{aligned}\frac{1}{3} =&\lambda p_1 x\\ \frac{2}{3} =&\lambda p_2 y\end{aligned}$$

Adding them and taking into account that $p_1x+p_2y=M$ we obtain that que $1=\lambda M$. Substituting

$$\lambda=\frac{1}{M}$$

into the fist two equations we obtain that

$$x=\frac{M}{3p_1},\quad y=\frac{2M}{3p_2}$$

5-16. *Find and classify the extreme points of the function f on the given set.*
(a) $f(x,y,z)=x^2+y^2+z^2$ *on the set* $\{(x,y,z)\in\mathbb{R}^3 : x+2y+z=1, 2x-3y-z=4\}$.
(b) $f(x,y,z)=(y+z-3)^2$ *on the set* $\{(x,y,z)\in\mathbb{R}^3 : x^2+y+z=2, x+y^2+2z=2\}$.
(c) $f(x,y,z)=x+y+z$ *on the set* $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1, x-y-z=1\}$.
(d) $f(x,y,z)=x^2+y^2+z^2$ *on the set* $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=z, x+y+z=4\}$.

Solution:
(a) The Lagrangian is

$$L(x,y,z)=f(x,y,z)-\lambda g(x,y,z)-\mu h(x,y,z)$$

The Lagrange equations are

$$\begin{aligned}D_x(L(x,y,z)) =&2x-\lambda-2\mu=0\\ D_y(L(x,y,z)) =&2y-2\lambda+3\mu=0\\ D_z(L(x,y,z)) =&2z-\lambda+\mu=0\\ &x+2y+z=1\\ &2x-3y-z=4\end{aligned}$$

whose solution is

$$x=\frac{92}{59}, y=-\frac{19}{59}, z=\frac{5}{59}, \mu=\frac{58}{59}, \lambda=\frac{68}{59}$$

(b) The Lagrangian is

$$L(x,y,z)=f(x,y,z)-\lambda g(x,y,z)-\mu h(x,y,z)$$

The Lagrange equations are

$$\begin{gathered}-2\lambda x-\mu=0\\ 2y+2z-6-\lambda-2\mu y=0\\ 2y+2z-6-\lambda-2\mu=0\\ x^2+y+z=2\\ x+y^2+2z=2\end{gathered}$$

From the second and third equations we obtain that

$$\mu=\mu y$$

and there are two possibilities: either $\mu=0$ or $y=1$.
If $\mu=0$, the first equation implies that $\lambda x=0$. One of factors must be 0. If $\lambda=0$, we obtain the equations,

$$\begin{gathered}2y+2z-6=0\\ x^2+y+z=2\\ x+y^2+2z=2\end{gathered}$$

But, the first equation implies $y+z=3$ and substituting in the second one one we obtain $x^2=-1$, which is impossible. If $x=0$, the last two equations are

$$\begin{aligned} y+z&=2 \\ y^2+2z&=2 \end{aligned}$$

But, now we obtain equation $y^2-2y+2=0$ which does not have real solutions.
We conclude that $\mu \neq 0$, so $y=1$. We may now use the last two equations to obtain x and z. The solutions to the Lagrange equations are

$$(\frac{-1}{2}, 1, \frac{3}{4}), \quad (1,1,0)$$

To see the character of these points, we apply the sufficient condition and calculate the second order derivatives of L with respect to the variables x, y, z, obtaining the Hessian

$$\mathcal{H}L_{x,y,z} = \begin{pmatrix} -2\lambda & 0 & 0 \\ 0 & 2-2\mu & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

We need the values of the multipliers associated to each point. Plugging the point P_1 into the Lagrange system above, we get $\lambda=\mu=-\frac{5}{6}$. The above Hessian becomes

$$\mathcal{H}L_{x,y,z} = \begin{pmatrix} \frac{5}{3} & 0 & 0 \\ 0 & \frac{11}{3} & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

which is positive definite, thus it is also positive definite when restricted to any subspace, in particular to the tangent subspace associated to P_1. Hence P_1 is a local minimum.
We do the same for the other point, P_2. The multipliers are in this case $\lambda=\frac{4}{3}$ and $\mu=-\frac{8}{3}$. The Hessian is

$$\mathcal{H}L_{x,y,z} = \begin{pmatrix} -\frac{8}{3} & 0 & 0 \\ 0 & \frac{22}{3} & 2 \\ 0 & 2 & 2 \end{pmatrix} \tag{1}$$

which is indefinite. We thus proceed to check the sign of the quadratic form in the tangent subspace to the constraints at the point P_2. We compute the gradients of the constraints at the point P_2. They are the vectors $(2x,1,1)$ and $(1,2y,2)$. At the point P_2 they become $(2,1,1)$ and $(1,2,2)$, respectively. The tangent space is the solution set of the system

$$\begin{aligned} 2x+y+z&=0 \\ x+2y+2z&=0. \end{aligned}$$

We find $\{(0,-z,z) : z \in \mathbb{R}\}$ as tangent space. Restricting the quadratic form (1) to this space, we get the restricted quadratic form $\frac{22}{3}z^2+2z^2-4z^2=\frac{16}{3}z^2$, which is positive definite. Thus, P_2 is a local minimum too.

Remark: *The above represents as far as the methods that we have studied in class will take us. If we want to determine if any of the above points corresponds to a global minimum, further analysis, beyond the scope of this course, is needed. We address this issue in this remark for those of you who are interested in the answer.*
From the first restrictions we obtain $z=2-x^2-y$. Replacing this value of z in the second restriction we get the equation

$$2=x+y^2+2z=x+y^2+4-2x^2-2y$$

so y satisfies the equation

$$y^2-2y+x-2x^2+2=0$$

Solving for y,

$$y=1\pm\sqrt{2x^2-x-1}$$

Thus, to obtain a solution of the two restrictions we need that $2x^2-x-1\geq 0$. Plotting the graph of the parabola $y=2x^2-x-1$ we see that $2x^2-x-1\geq 0$ for $x\leq -1/2$ and $x\geq 1$. That is, if we let $I=(-\infty-\frac{1}{2}]\cup[1,\infty)$, we may parametrize the set $\{(x,y,z)\in\mathbb{R}^3 : x^2+y+z=2, x+y^2+2z=2\}$ as

$$\{(x,y,z)\in\mathbb{R}^3 : x\in I, y=1\pm\sqrt{2x^2-x-1}, z=2-x^2-y,\}$$

Now, replacing $y+z=2-x^2$ in the objective function we obtain $f(x,y,z)=(y+z-3)^2=(1+x^2)^2$. Note that function of one variable $h(x)=(1+x^2)^2$, defined in the domain I, is strictly convex, decreasing for $x\le 0$ and increasing for $x\ge 0$. Thus, there are two local minima at the points $-1/2$ and 1. Furthermore,

$$h\left(-\frac{1}{2}\right)=\frac{1}{4},\quad h(1)=4$$

Thus, the global minimum is attained at the point $x=-\frac{1}{2}$, $y=1$, $z=\frac{3}{4}$.

(c) The Lagrangian is $L(x,y,z)=f(x,y,z)-\lambda g(x,y,z)-\mu h(x,y,z)$ The Lagrange equations are

$$\begin{gathered}1-2\lambda x-\mu=0\\ 1-2\lambda y+\mu=0\\ 1-2\lambda z+\mu=0\\ x^2+y^2+z^2=1\\ x-y-z=1\end{gathered}$$

From the second and third equations a we obtain that $\lambda y=\lambda z$.
If $\lambda=0$, the first two equations are inconsistent. Therefore, $\lambda\neq 0$ and $y=z$. Now, we may use the last two equations to solve for y and z. We obtain the solutions

$$\begin{gathered}(1,0,0)\ \text{maximum}\\ -\frac{1}{3}(1,2,2)\ \text{minimum}\end{gathered}$$

5-17. *Find the maximum of the function $f(x,y,z)=xyz$ on the set $\{(x,y,z)\in\mathbb{R}^3 : x+y+z\le 1,\quad x,y,z\ge 0\}$.*

Solution: First we write the problem in the canonical form,

$$\begin{array}{ll}\max\limits_{x,y,z} & xyz\\ \text{s.t.} & x+y+z\le 1\\ & -x\le 0\\ & -y\le 0\\ & -z\le 0\end{array}$$

The Lagrangian is

$$L=xyz+\lambda_1(1-x-y-z)+\lambda_2 x+\lambda_3 y+\lambda_4 z$$

and the Kuhn–Tucker equations are

$$\frac{\partial L}{\partial x}=yz-\lambda_1+\lambda_2 = 0\Leftrightarrow\lambda_1=yz+\lambda_2 \tag{2}$$

$$\frac{\partial L}{\partial y}=xz-\lambda_1+\lambda_3 = 0\Leftrightarrow\lambda_1=xz+\lambda_3 \tag{3}$$

$$\frac{\partial L}{\partial z}=xy-\lambda_1+\lambda_4 = 0\Leftrightarrow\lambda_1=xy+\lambda_4 \tag{4}$$

$$\lambda_1(1-x-y-z) = 0 \tag{5}$$

$$\lambda_2 x = 0 \tag{6}$$

$$\lambda_3 y = 0 \tag{7}$$

$$\lambda_4 z = 0 \tag{8}$$

$$\lambda_1,\lambda_2,\lambda_3 \ge 0 \tag{9}$$

$$x+y+z \le 1 \tag{10}$$

$$x,y,z \ge 0 \tag{11}$$

Case 1 $\lambda_1=0$**.** In this case, from equations 2, 3 y 4 we obtain that

$$yz+\lambda_2=xz+\lambda_3=xy+\lambda_4=0$$

But all the variables are positive, so

$$yz=xz=xy=\lambda_2=\lambda_3=\lambda_4=0 \tag{12}$$

The equations 12 yield an infinite number of solutions in which at least two of the variables x, y, z vanish. The value of the objective function at these solutions is 0.

Case 2 $\lambda_1 > 0$. Then, $x + y + z = 1$. If, for example $x = 0$, we obtain from equations 3 y 4 that

$$\lambda_3 = \lambda_4 = \lambda_1 > 0$$

and from the equations 7 y 8 we see that $y = z = 0$. But this contradicts that $x + y + z = 1$. We conclude that $x > 0$. Analogously, one can show that $y, z > 0$. By equations 6, 7 y 8 we see that

$$\lambda_2 = \lambda_3 = \lambda_4 = 0$$

and equations 2, 3 y 4 imply that

$$yz = xz = xy$$

that is

$$x = y = z$$

And since $x + y + z = 1$, we conclude that

$$x = y = z = \frac{1}{3}; \quad \lambda_1 = \frac{1}{9}$$

The value of the objective function is

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}$$

So the point

$$x = y = z = \frac{1}{3}$$

corresponds to the maximum.

5-18. *Find the minimum of the function* $f(x, y) = 2y - x^2$ *on the set* $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1, \quad x, y \ge 0\}$.

Solution: First we write the problem in the canonical form,

$$\begin{array}{ll} \max\limits_{x,y,z} & x^2 - 2y \\ \text{s.t.} & x^2 + y^2 \le 1 \\ & -x \le 0 \\ & -y \le 0 \end{array}$$

The Lagrangian is

$$L = x^2 - 2y + \lambda_1(1 - x^2 - y^2) + \lambda_2 x + \lambda_3 y$$

and the Kuhn–Tucker equations are

$$\frac{\partial L}{\partial x} = 2x - 2\lambda_1 x + \lambda_2 = 0 \Leftrightarrow 2\lambda_1 x = 2x + \lambda_2 \tag{13}$$

$$\frac{\partial L}{\partial y} = -2 - 2\lambda_1 y + \lambda_3 = 0 \Leftrightarrow \lambda_3 = 2 + 2\lambda_1 y \tag{14}$$

$$\lambda_1(1 - x^2 - y^2) = 0 \tag{15}$$

$$\lambda_2 x = 0 \tag{16}$$

$$\lambda_3 y = 0 \tag{17}$$

$$\lambda_1, \lambda_2, \lambda_3 \ge 0 \tag{18}$$

$$x^2 + y^2 \le 1 \tag{19}$$

$$x, y \ge 0 \tag{20}$$

Since all the variables that appear in equation 14 are positive, we have that

$$\lambda_3 \ge 2 > 0$$

so

$$y = 0$$

and from equation 14 we see that

$$\lambda_3 = 2$$

Case 1: $x = 0$. By equation 15 we see that $\lambda_1 = 0$ y by equation 13 we see that $\lambda_2 = 0$. That is, of solution of the Kuhn–Tucker equations is

$$x^* y^* = 0; \quad \lambda_1 = \lambda_2 = 0, \quad \lambda_3 = 2$$

The value of the objective function at these solutions is $f(0,0)=0$.

Case 2: $x>0$**.** Then, $\lambda_2=0$ and by equation 13, $\lambda_1=1$. Now we obtain from equation 15 that $x^2+y^2=1$. And since $y=0$, $x\geq 0$ we have that

$$x=1$$

Another solution of the Kuhn–Tucker equations is

$$x^*=1,\quad y^*=0;\quad \lambda_1=1,\quad \lambda_2=0,\quad \lambda_3=2$$

The value of the objective function at these solutions is $f(1,0)=-1$. Therefore, the minimum is attained at the point $(1,0)$.

5-19. *Solve the optimization problem*

$$\begin{cases} \min & x^2+y^2-20x \\ s.t. & 25x^2+4y^2\leq 100 \end{cases}$$

Solution: We write the problem as

$$\begin{cases} \max & -x^2-y^2+20x \\ \text{s.t.} & 25x^2+4y^2\leq 100 \end{cases}$$

The associated Lagrangian is

$$L=-x^2-y^2+20x+\lambda(100-25x^2-4y^2)$$

and the Kuhn-Tucker equations are

$$\begin{aligned} -2x+20-50\lambda x &= 0 \quad (21)\\ -2y-8\lambda y &= 0 \quad (22)\\ 25x^2+4y^2 &\leq 100 \quad (23)\\ \lambda(100-25x^2-4y^2) &= 0 \quad (24)\\ \lambda &\geq 0 \quad (25) \end{aligned}$$

From 21 we see that $y(1+4\lambda)=0$, so if $y\neq 0$, then $\lambda=-1/4$ which contradicts 25. We conclude that $y=0$ and the system reduces to

$$\begin{aligned} -2x+20-50\lambda x &= 0 \quad (26)\\ x^2 &\leq 4 \quad (27)\\ \lambda(4-x^2) &= 0 \quad (28)\\ \lambda &\geq 0 \quad (29) \end{aligned}$$

If $\lambda=0$ from 26 we obtain that $x=10$. But this does not satisfy 27. Therefore, $\lambda\neq 0$ and from 28 we see that $x^2=4$. There are two possibilities $x=\pm 2$. Solving in 26 we obtain

$$\lambda=\frac{10-x}{25x}$$

so that if $x=-2$ then $\lambda=-12/50$ does not satisfy 29. Therefore, the solution is

$$x=2,\quad y=0,\qquad \lambda=\frac{8}{50}$$

5-20. *Solve the optimization problem*

$$\begin{cases} \max & x+y-2z \\ s.t. & z\geq x^2+y^2 \\ & x,y,z\geq 0 \end{cases}$$

Solution: First we write the problem in the canonical form,

$$\begin{array}{ll} \max\limits_{x,y,z} & x+y-2z \\ \text{s.t.} & x^2+y^2-z \leq 0 \\ & -x \leq 0 \\ & -y \leq 0 \\ & -z \leq 0 \end{array}$$

The Lagrangian is

$$L = x + y - 2z + \lambda_1(z - x^2 - y^2) + \lambda_2 x + \lambda_3 y + \lambda_4 z$$

and the Kuhn–Tucker equations are

$$\frac{\partial L}{\partial x} = 1 - 2\lambda_1 x + \lambda_2 = 0 \Leftrightarrow 2\lambda_1 x = 1 + \lambda_2 \tag{30}$$

$$\frac{\partial L}{\partial y} = 1 - 2\lambda_1 y + \lambda_3 = 0 \Leftrightarrow 2\lambda_1 y = 1 + \lambda_3 \tag{31}$$

$$\frac{\partial L}{\partial z} = -2 + \lambda_1 + \lambda_4 = 0 \Leftrightarrow \lambda_1 + \lambda_4 = 2 \tag{32}$$

$$\lambda_1(x^2 + y^2 - z) = 0 \tag{33}$$

$$\lambda_2 x = 0 \tag{34}$$

$$\lambda_3 y = 0 \tag{35}$$

$$\lambda_4 z = 0 \tag{36}$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \tag{37}$$

$$x, y, z \geq 0 \tag{38}$$

From equation 30, we see that if $\lambda_1 = 0$ o $x = 0$, then $1 + \lambda_2 = 0$ so $\lambda_2 < 0$. Therefore, we must have that $\lambda_1 > 0$ y $x > 0$. Similarly, from equation 31 we obtain that $y > 0$. From equations 34 and 35 we see now that $\lambda_2 = \lambda_3 = 0$. The Kuhn–Tucker equations may now be written as follows

$$2\lambda_1 x = 1 \tag{39}$$

$$2\lambda_1 y = 1 \tag{40}$$

$$\lambda_1 + \lambda_4 = 2 \tag{41}$$

$$x^2 + y^2 = z \tag{42}$$

$$\lambda_4 z = 0 \tag{43}$$

$$\lambda_1, \lambda_4 \geq 0 \tag{44}$$

$$x, y, z \geq 0 \tag{45}$$

Since $\lambda_1 > 0$, from equations 39 and 40 we obtain that

$$x = y = \frac{1}{2\lambda_1}$$

so

$$z = 2x^2 > 0$$

And we deduce from equation 43 that $\lambda_4 = 0$ and from equation 41 that $\lambda_1 = 2$. Therefore, the solution is

$$x = y = \frac{1}{4}, \quad z = \frac{1}{8}; \qquad \lambda_1 = 2, \quad \lambda_2 = \lambda_3 = \lambda_4 = 0$$