CHAPTER 4: Higher order derivatives

4-1. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be defined by $u(x,y) = e^x \sin y$. Find all the second partial derivatives D^2u, and verify Schwarz's Theorem.*

Solution: The partial derivatives of the function $u(x,y) = e^x \sin y$ are

$$\frac{\partial u}{\partial x} = e^x \sin y, \qquad \frac{\partial u}{\partial y} = e^x \cos y$$

Therefore the Hessian is

$$\begin{pmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{pmatrix}$$

4-2. *Consider the quadratic function $Q : \mathbb{R}^3 \to \mathbb{R}$ defined by $Q(x,y,z) = x^2 + 5y^2 + 4xy - 2yz$. Compute the Hessian matrix D^2Q.*

Solution: The gradient of Q is

$$\nabla(x^2 + 5y^2 + 4xy - 2yz) = (2x + 4y, 10y + 4x - 2z, -2y)$$

The Hessian matrix of Q is

$$\begin{pmatrix} 2 & 4 & 0 \\ 4 & 10 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

4-3. *Let $f(x,y,z) = e^z + \frac{1}{x} + xe^{-y}$, for $x \neq 0$. Compute*

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}$$

Solution: The partial derivatives of the function $f(x,y,z) = e^z + \frac{1}{x} + xe^{-y}$ are

$$\begin{aligned}
\frac{\partial^2 f(x,y,z)}{\partial x^2} &= \frac{2}{x^3} \\
\frac{\partial^2 f(x,y,z)}{\partial x \partial y} &= -\, e^{-y} \\
\frac{\partial^2 f(x,y,z)}{\partial y \partial x} &= -\, e^{-y} \\
\frac{\partial^2 f(x,y,z)}{\partial y^2} &= xe^{-y}
\end{aligned}$$

4-4. *Let $z = f(x,y)$, $x = at$, $y = bt$ where a and b are constant. Consider z as a function of t. Compute $\frac{d^2 z}{dt^2}$ in terms of a, b and the second partial derivatives of f: f_{xx}, f_{yy} and f_{xy}.*

Solution: Since the function is of class C^2, we may apply Schwarz's Theorem.

$$\begin{aligned}
\frac{d}{dt}(f(at,bt)) &= af_x\,(at,bt) + bf_y\,(at,bt) \\
\frac{d^2}{dt^2}(f(at,bt)) &= a^2 f_{xx}\,(at,bt) + 2ab f_{xy}\,(at,bt) + b^2 f_{yy}\,(at,bt)
\end{aligned}$$

4-5. *Let* $f(x,y) = 3x^2y + 4x^3y^4 - 7x^9y^4$. *Compute the Hessian matrix* D^2Q..

Solution: The gradient of f is

$$\nabla f(x,y) = \left(6xy + 12x^2y^4 - 63x^8y^4, 3x^2 + 16x^3y^3 - 28x^9y^3\right)$$

The Hessian matrix of f is

$$\mathrm{H}(x,y) = \begin{pmatrix} 6y + 24xy^4 - 504x^7y^4 & 6x + 48x^2y^3 - 252x^8y^3 \\ 6x + 48x^2y^3 - 252x^8y^3 & 48x^3y^2 - 74x^9y^2 \end{pmatrix}$$

4-6. *Let* $f, g : \mathbb{R}^2 \to \mathbb{R}$ *be two functions whose partial derivatives are continuous on all of* $\mathbb{R}^2$ *and such that there is a function n* $h : \mathbb{R}^2 \to \mathbb{R}$ *such that* $(f,g) = \nabla h$, *that is,*

$$f(x,y) = \frac{\partial h}{\partial x}(x,y) \quad g(x,y) = \frac{\partial h}{\partial y}(x,y)$$

at every point $(x,y) \in \mathbb{R}^2$. *What equation do*

$$\frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial g}{\partial x}$$

satisfy?

Solution: On the one hand, we have that

$$\frac{\partial f}{\partial y} = \frac{\partial\left(\frac{\partial h}{\partial x}\right)}{\partial y} = \frac{\partial^2 h}{\partial x \partial y}$$

On the other hand, we see that

$$\frac{\partial g}{\partial x} = \frac{\partial\left(\frac{\partial h}{\partial y}\right)}{\partial x} = \frac{\partial^2 h}{\partial y \partial x}$$

Since the functions f and g have continuous partial derivatives on all of $\mathbb{R}^2$, the function h is of class C^2. By Schwartz's Theorem, we conclude that

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x}$$

That is,

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

4-7. *The demand function of a consumer by a system of equations of the form*

$$\begin{aligned} \frac{\partial u}{\partial x} &= \lambda p_1 \\ \frac{\partial u}{\partial y} &= \lambda p_2 \\ p_1 x + p_2 y &= m \end{aligned}$$

where $u(x,y)$ *is the utility function of the agent,* p_1 *and* p_2 *are th prices of the consumption bundles,* m *is income and* $\lambda \in \mathbb{R}$. *Assuming that this system determines* x, y *and* λ *as functions of the other parameters, determine*

$$\frac{\partial x}{\partial p_1}$$

Solution: First we write the system as

$$\begin{aligned} f_1 &\equiv \frac{\partial u}{\partial x} - \lambda p_1 = 0 \\ f_2 &\equiv \frac{\partial u}{\partial y} - \lambda p_2 = 0 \\ f_3 &\equiv p_1 x + p_2 y - m = 0 \end{aligned}$$

and compute

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, \lambda)} = \begin{vmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & -p_1 \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} & -p_2 \\ p_1 & p_2 & 0 \end{vmatrix} = \frac{\partial^2 u}{\partial x^2} p_2^2 - \frac{\partial^2 u}{\partial x \partial y} p_1 p_2 + \frac{\partial^2 u}{\partial y^2} p_1^2$$

We suppose that this determinant does not vanish and that we may apply the mean value Theorem. Differentiating with respect to p_1 (but assuming now that x, y, λ depend on the other parameters) we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial p_1} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial p_1} - \frac{\partial \lambda}{\partial p_1}p_1 - \lambda &= 0 \\ \frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial p_1} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial p_1} - \frac{\partial \lambda}{\partial p_1}p_2 &= 0 \\ x + p_1\frac{\partial x}{\partial p_1} + p_2\frac{\partial y}{\partial p_1} = 0 \end{aligned}$$

which may be written as

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial p_1} &+ \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial p_1} - \frac{\partial \lambda}{\partial p_1}p_1 = \lambda \\ \frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial p_1} &+ \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial p_1} - \frac{\partial \lambda}{\partial p_1}p_2 = 0 \\ p_1\frac{\partial x}{\partial p_1} &+ p_2\frac{\partial y}{\partial p_1} = -x \end{aligned}$$

The unknowns of the above system are

$$\frac{\partial x}{\partial p_1}, \quad \frac{\partial y}{\partial p_1}, \quad \frac{\partial \lambda}{\partial p_1}$$

We see that the determinant of the system is

$$\frac{\partial (f_1, f_2, f_3)}{\partial (x, y, \lambda)}$$

Using Cramer's rule we see that,

$$\frac{\partial x}{\partial p_1} = \frac{\begin{vmatrix} \lambda & \frac{\partial^2 u}{\partial x \partial y} & -p_1 \\ 0 & \frac{\partial^2 u}{\partial y^2} & -p_2 \\ -x & p_2 & 0 \end{vmatrix}}{\frac{\partial^2 u}{\partial x^2}p_2^2 - \frac{\partial^2 u}{\partial x \partial y}p_1 p_2 + \frac{\partial^2 u}{\partial y^2}p_1^2} = \frac{\lambda p_2^2 + m\frac{\partial^2 u}{\partial x \partial y}p_2 - m\frac{\partial^2 u}{\partial y^2}p_1}{\frac{\partial^2 u}{\partial x^2}p_2^2 - \frac{\partial^2 u}{\partial x \partial y}p_1 p_2 + \frac{\partial^2 u}{\partial y^2}p_1^2}$$

4-8. *Consider the system of equations*

$$\begin{aligned} z^2 + t - xy &= 0 \\ zt + x^2 &= y^2 \end{aligned}$$

(a) *Prove that it determines z and t as functions of x, y near the point $(1, 0, 1, -1)$.*
(b) *Compute the partial derivatives of z and t with respect to x, y at $(1, 0)$.*
(c) *Without solving the system, ¿what is approximate value of $z(1'001, 0'002)$*
(d) *Compute*

$$\frac{\partial^2 z}{\partial x \partial y}(1, 0)$$

Solution:

(a) First we write the system as

$$\begin{aligned} f_1 &\equiv z^2 + t - xy = 0 \\ f_2 &\equiv zt + x^2 - y^2 = 0 \end{aligned}$$

and compute

$$\frac{\partial (f_1, f_2)}{\partial (z, t)} = \begin{vmatrix} 2z & 1 \\ t & z \end{vmatrix} = 2z^2 - t$$

which does not vanish for $z = 1$, $t = -1$. Therefore, we may apply the implicit function Theorem. Differentiating the above system with respect to x we obtain

$$\begin{aligned} 2z\frac{\partial z}{\partial x} + \frac{\partial t}{\partial x} - y &= 0 \\ t\frac{\partial z}{\partial x} + z\frac{\partial t}{\partial x} + 2x &= 0 \end{aligned}$$

Now we plug in the values $x = 1$, $y = 0$, $z = 1$, $t = -1$ and obtain

$$\begin{aligned} 2\frac{\partial z}{\partial x}(1,0) &+ \frac{\partial t}{\partial x}(1,0) = 0 \\ -\frac{\partial z}{\partial x}(1,0) &+ \frac{\partial t}{\partial x}(1,0) = -2 \end{aligned}$$

so

$$\frac{\partial z}{\partial x}(1,0) = \frac{2}{3}, \quad \frac{\partial t}{\partial x} = -\frac{4}{3}$$

Differentiating the above system with respect to y we obtain

(1)
$$\begin{aligned} 2z\frac{\partial z}{\partial y} + \frac{\partial t}{\partial y} - x &= 0 \\ t\frac{\partial z}{\partial y} + z\frac{\partial t}{\partial y} - 2y &= 0 \end{aligned}$$

Now we plug in the values $x = 1$, $y = 0$, $z = 1$, $t = -1$ and obtain

$$\begin{aligned} 2\frac{\partial z}{\partial y}(1,0) + \frac{\partial t}{\partial y}(1,0) &= 1 \\ -\frac{\partial z}{\partial y}(1,0) + \frac{\partial t}{\partial y}(1,0) &= 0 \end{aligned}$$

so

$$\frac{\partial z}{\partial y}(1,0) = \frac{1}{3}, \quad \frac{\partial t}{\partial y}(1,0) = \frac{1}{3}$$

(b) We use Taylor's first order approximation about the point $(1,0)$

$$P_1(x,y) = z(1,0) + \frac{\partial z}{\partial x}(1,0)(x-1) + \frac{\partial z}{\partial y}(1,0)y = 1 + \frac{2}{3}(x-1) + \frac{y}{3}$$

and we obtain that

$$z(1'001, 0'002) \approx P(1'001, 0'002) = 1 + \frac{0'002}{3} + \frac{0'002}{3} = 1'00133$$

(c) Differentiating implicitly the system 1 with respect to x,

$$\begin{aligned} 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + 2z\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 t}{\partial x \partial y} - 1 &= 0 \\ \frac{\partial t}{\partial x}\frac{\partial z}{\partial y} + t\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x}\frac{\partial t}{\partial y} + z\frac{\partial^2 t}{\partial x \partial y} &= 0 \end{aligned}$$

Now we plug in the values

$$x = 1, \quad y = 0, \quad z = 1, \quad t = -1, \quad \frac{\partial z}{\partial x}(1,0) = \frac{2}{3}, \quad \frac{\partial t}{\partial x} = -\frac{4}{3}, \quad \frac{\partial z}{\partial y}(1,0) = \frac{1}{3}, \quad \frac{\partial t}{\partial y} = \frac{1}{3}$$

so the system becomes

$$\begin{aligned} 2\frac{\partial^2 z}{\partial x \partial y}(1,0) + \frac{\partial^2 t}{\partial x \partial y}(1,0) &= \frac{5}{9} \\ -\frac{\partial^2 z}{\partial x \partial y}(1,0)\frac{\partial^2 t}{\partial x \partial y}(1,0) &= \frac{2}{9} \end{aligned}$$

and solving it we obtain that

$$\frac{\partial^2 t}{\partial x \partial y}(1,0) = \frac{3}{9}, \quad \frac{\partial^2 z}{\partial x \partial y}(1,0) = \frac{1}{9}$$

4-9. *Consider the system of equations*

$$\begin{aligned} xt^3 + z - y^2 &= 0 \\ 4zt &= x - 4 \end{aligned}$$

(a) *Prove that it determines z and t as functions of x, y near the point $(0, 1, 1, -1)$.*
(b) *Compute the partial derivatives of z and t with respect to x, y at $(0, 1)$.*
(c) *Without solving the system, ¿what is approximate value of $z(0'001, 1'002)$*

(d) *Compute*

$$\frac{\partial^2 z}{\partial x \partial y}(0,1)$$

Solution:

(a) First we write the system as

$$\begin{aligned} f_1 &\equiv xt^3 + z - y^2 = 0 \\ f_2 &\equiv 4zt - x + 4 = 0 \end{aligned}$$

and compute

$$\frac{\partial(f_1, f_2)}{\partial(z,t)} = \begin{vmatrix} 1 & 3xt^2 \\ 4t & 4z \end{vmatrix} = 4z - 12xt^3$$

which does not vanish for $x = 0, y = 1, z = 1, t = -1$. Therefore, we may apply the implicit function Theorem.

Differentiating the above system with respect to x we obtain

$$\begin{aligned} t^3 + 3xt^2\frac{\partial t}{\partial x} + \frac{\partial z}{\partial x} &= 0 \\ 4t\frac{\partial z}{\partial x} + 4z\frac{\partial t}{\partial x} - 1 &= 0 \end{aligned}$$

Now we plug in the values $x = 0, y = 1, z = 1, t = -1$ and obtain

$$\begin{aligned} -1 + \frac{\partial z}{\partial x}(0,1) &= 0 \\ -4\frac{\partial z}{\partial x}(0,1) + 4\frac{\partial t}{\partial x}(0,1) - 1 &= 0 \end{aligned}$$

so

$$\frac{\partial z}{\partial x}(0,1) = 1, \quad \frac{\partial t}{\partial x}(0,1) = \frac{5}{4}$$

Differentiating the above system with respect to y we obtain

$$\begin{aligned} 3xt^2\frac{\partial t}{\partial y} + \frac{\partial z}{\partial y} - 2y &= 0 \\ t\frac{\partial z}{\partial y} + z\frac{\partial t}{\partial y} &= 0 \end{aligned} \tag{2}$$

Now we plug in the values $x = 0, y = 1, z = 1, t = -1$ and obtain

$$\begin{aligned} \frac{\partial z}{\partial y}(0,1) - 2 &= 0 \\ -\frac{\partial z}{\partial y}(0,1) + \frac{\partial t}{\partial y}(0,1) &= 0 \end{aligned}$$

so

$$\frac{\partial z}{\partial y}(0,1) = 2, \quad \frac{\partial t}{\partial y}(0,1) = 2$$

(b) We use Taylor's first order approximation about the point $(0,1)$,

$$P_1(x,y) = z(0,1) + \frac{\partial z}{\partial x}(0,1)x + \frac{\partial z}{\partial y}(0,1)(y-1) = x + 2y - 1$$

and we obtain that

$$z(0'001, 1'002) \approx P(0'001, 1'002) = 0'001 + 2'004 - 1 = 1'005$$

(c) Differentiating implicitly the system 2 with respect to x,

$$\begin{aligned} 3t^2\frac{\partial t}{\partial y} + 6xt\frac{\partial t}{\partial x}\frac{\partial t}{\partial y} + 3xt^2\frac{\partial^2 t}{\partial x\partial y} + \frac{\partial^2 z}{\partial x\partial y} &= 0 \\ 4\frac{\partial t}{\partial x}\frac{\partial z}{\partial y} + 4t\frac{\partial^2 z}{\partial x\partial y} + 4\frac{\partial z}{\partial x}\frac{\partial t}{\partial y} + 4z\frac{\partial^2 t}{\partial x\partial y} &= 0 \end{aligned}$$

Now we plug in the values

$$x = 0, y = 1, z = 1, t = -1, \quad \frac{\partial z}{\partial x}(0,1) = 1, \quad \frac{\partial t}{\partial x}(0,1) = \frac{5}{4}, \quad \frac{\partial z}{\partial y}(0,1) = 2, \quad \frac{\partial t}{\partial y}(0,1) = 2$$

so the system becomes

$$\begin{aligned} 6+\frac{\partial^2 z}{\partial x\partial y}(0,1) &= 0\\ 18-4\frac{\partial^2 z}{\partial x\partial y}+4\frac{\partial^2 t}{\partial x\partial y} &= 0 \end{aligned}$$

and solving it we obtain that

$$\frac{\partial^2 t}{\partial x\partial y}(0,1)=-\frac{21}{2},\quad \frac{\partial^2 z}{\partial x\partial y}(0,1)=-6$$

4-10. *Find the second order Taylor polinomial for the following functions about the given point.*
(a) $f(x,y)=\ln(1+x+2y)$ *about the point* $(2,1)$.
(b) $f(x,y)=x^3+3x^2y+6xy^2-5x^2+3xy^2$ *about the point* $(1,2)$.
(c) $f(x,y)=e^{x+y}$ *about the point* $(0,0)$.
(d) $f(x,y)=\sin(xy)+\cos(xy)$ *about the point* $(0,0)$.
(e) $f(x,y,z)=x-y^2+xz$ *about the point* $(1,0,3)$.

Solution: The Taylor polynomial of order 2 of f around the point x_0 is

$$P_2(x)=f(x_0)+\nabla f(x_0)\cdot(x-x_0)+\frac{1}{2!}(x-x_0)^tHf(x_0)(x-x_0)$$

(a) $f(x,y)=\log(1+x+2y)$ in $(2,1)$. Since,

$$\nabla f(2,1)=\left(\frac{1}{1+x+2y},\frac{2}{1+x+2y}\right)\Big|_{x=2,y=1}=\left(\frac{1}{5},\frac{2}{5}\right)$$

the Hessian is

$$\begin{pmatrix} -\frac{1}{(1+x+2y)^2} & -\frac{2}{(1+x+2y)^2}\\ -\frac{2}{(1+x+2y)^2} & -\frac{4}{(1+x+2y)^2}\end{pmatrix}\Bigg|_{x=2,y=1}=\begin{pmatrix} -\frac{1}{25} & -\frac{2}{25}\\ -\frac{2}{25} & -\frac{4}{25}\end{pmatrix}$$

and

$$f(2,1)=\ln 5$$

we have that Taylor's polynomial is

$$P_2(x)=\ln 5+\frac{1}{5}(x-2)+\frac{2}{5}(y-1)+-\frac{1}{50}(x-2)^2-\frac{2}{25}(x-2)(y-1)-\frac{2}{25}(y-1)^2$$

(b) $f(x,y)=x^3+3x^2y+6xy^2-5x^2+3xy^2$ in $(1,2)$. Since,

$$\nabla f(1,2)=\left(3x^2+6yx+9y^2-10x,3x^2+18yx\right)\big|_{x=1,y=2}=(41,39)$$

the Hessian is

$$\begin{pmatrix} 6x+6y-10 & 6x+18y\\ 6x+18y & 18x\end{pmatrix}\Bigg|_{x=1,y=2}=\begin{pmatrix} 8 & 42\\ 42 & 18\end{pmatrix}$$

and

$$f(1,2)=38$$

we have that Taylor's polynomial is

$$P_2(x)=38+41(x-1)+39(y-2)+4(x-1)^2+42(x-1)(y-2)+9(y-2)^2$$

(c) $f(x,y)=e^{x+y}$ at the point $(0,0)$. We have that $f(0,0)=1$. The gradient is

$$\nabla f(0,0)=\left(e^{x+y},e^{x+y}\right)\big|_{x=0,y=0}=(e^0,e^0)]$$

and the Hessian is

$$\begin{pmatrix} e^{x+y} & e^{x+y}\\ e^{x+y} & e^{x+y}\end{pmatrix}\Bigg|_{x=0,y=0}=\begin{pmatrix} e^0 & e^0\\ e^0 & e^0\end{pmatrix}$$

Thus, Taylor's polynomial is

$$P_2(x)=1+1x+1y+\frac{1}{2!}(1x^2+2xy+y^2)=1+x+y+\frac{1}{2}x^2+yx+\frac{1}{2}y^2$$

(d) $f(x,y)=\sin(xy)+\cos(xy)$ at the point $(0,0)$. First, $f(0,0)=1$. The gradient is

$$\nabla f(0,0) = \left((\cos yx)\, y - (\sin yx)\, y, (\cos yx)\, x - (\sin yx)\, x\right)|_{x=0,y=0} = (0,0)$$

The second derivatives are

$$\frac{\partial^2 f}{(\partial x)^2} = -y^2 \sin(yx) - y^2\cos(yx)$$
$$\frac{\partial^2 f}{\partial x \partial y} = -yx\sin(yx) + \cos(yx) - yx\cos(yx) - \sin(yx)$$
$$\frac{\partial^2 f}{(\partial y)^2} = x^2 - x^2\cos(yx)$$

and Hessian at the point $(0,0)$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence, Taylor's polynomial is

$$P_2(x,y) = 1 + yx$$

(e) $f(x,y,z)=x-y^2+xz$ at the point $(1,0,3)$. First, $f(1,0,3)=4$. The gradient is

$$\nabla f(1,0,3) = \left(1+z, -2y, x\right)|_{x=1,y=0,z=3} = (4,0,1)$$

and the Hessian is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Hence, Taylor's polynomial is

$$P_2(x,y,z) = 4 + 4(x-1) + (z-3) + \frac{1}{2!}(-2y^2 + 2(x-1)(z-3))$$

4-11. *For what values of the parameter a is the quadratic form $Q(x,y,z) = x^2-2axy-2xz+y^2+4yz+5z^2$ positive definite?*

Solution: $Q(x,y,z) = x^2-2axy-2xz+y^2+4yz+5z^2$
It will be positive definite if $D_1>0, D_2>0, D_3>0$. Let us compute these.
$D_1 = 1$
$D_2 = \begin{vmatrix} 1 & -a \\ -a & 1 \end{vmatrix} = 1-a^2>0$ if and only if $|a|<1$.
$D_3 = \begin{vmatrix} 1 & -a & -1 \\ -a & 1 & 2 \\ -1 & 2 & 5 \end{vmatrix} = -5a^2+4a = a(4-5a)>0$ if and only if $a\in(0,4/5)$.

Therefore, the quadratic form is positive definite if $a\in(0,4/5)$. When $a=0$ or $a=4/5$, we have that $D_1>0$, $D_2>0$, $D_3=0$. So, the quadratic form is positive semidefinite, but not positive definite. When $a\in(-\infty,0)\cup(\frac{4}{5},+\infty)$ we see that $D_1>0$, $D_3<0$ so the quadratic form is indefinite.

4-12. *Study the signature of the following quadratic forms.*
(a) $Q_1(x,y,z) = x^2+7y^2+8z^2-6xy+4xz-10yz$.
(b) $Q_2(x,y,z) = -2y^2-z^2+2xy+2xz+4yz$.

Solution: a) The matrix associated to Q_1 is $\begin{pmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{pmatrix}$. Let us compute $D_1=1>0$, $D_2 = \begin{vmatrix} 1 & -3 \\ -3 & 7 \end{vmatrix} = -2$ and $D_3 = \begin{vmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{vmatrix} = -9$. Therefore, the quadratic form is indefinite. (Note that it was not necessary to compute D_3)

b) The matrix associated to Q_2 is $\begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{pmatrix}$. We see that $D_1=0$. Can we still apply the method of principal minors? To do so we perform the following change of variables: $\bar{x}=z$, $\bar{z}=x$. We see that

$$Q_2(\bar{x},y,\bar{z}) = -2y^2-\bar{x}^2+2\bar{z}y+2\bar{x}\bar{z}+4y\bar{x}$$

whose associated matrix is $\begin{pmatrix} -1 & 2 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. The principal minors are $D_1 = -1$, $D_2 = \begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix} = -2$. Therefore, the quadratic form is indefinite.

Here is another way to do this exercise. Since, $D_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 7 \neq 0$. But, $D_1 = 0$, $D_2 = -1$, so by Proposition 3.13, the quadratic form is indefinite.

4-13. *Study for what values of a the quadratic form $Q(x, y, z) = ax^2 + 4ay^2 + 4az^2 + 4xy + 2axz + 4yz$ is*
(a) *positive definite.*
(b) *negative definite.*

Solution: The matrix associated to the quadratic form $Q(x, y, z) = ax^2 + 4ay^2 + 4az^2 + 4xy + 2axz + 4yz$ is

$$\begin{pmatrix} a & 2 & a \\ 2 & 4a & 2 \\ a & 2 & 4a \end{pmatrix}$$

(a) We study conditions under which the principal minors satisfy the following
 (i) $D_1 = a > 0$.
 (ii) $D_2 = \begin{vmatrix} a & 2 \\ 2 & 4a \end{vmatrix} = 4a^2 - 4 = 4(a^2 - 1) > 0$. This condition is satisfied if and only if $|a| > 1$
 (iii) $D_3 = \begin{vmatrix} a & 2 & a \\ 2 & 4a & 2 \\ a & 2 & 4a \end{vmatrix} = 12a^3 - 12a = 12a(a^2 - 1) > 0$.
 Assuming $a > 0$, the condition $a(a^2 - 1) > 0$ simplifies to $(a^2 - 1) > 0$ which is satisfied if and only if $|a| > 1$. Therefore, Q es positive definite if $a > 1$.
(b) We study conditions under which the principal minors satisfy the following
 (i) $D_1 = a < 0$.
 (ii) $D_2 = \begin{vmatrix} a & 2 \\ 2 & 4a \end{vmatrix} = 4a^2 - 4 = 4(a^2 - 1) > 0$ This condition is satisfied if and only if $|a| > 1$.
 Assuming, $a < 0$, the equation $4(a^2 - 1) > 0$ implies that $a < -1$. In the previous part we have seen that $D_3 = 12a(a^2 - 1) < 0$ if $a < -1$. Therefore, Q is definite negative if $a < -1$.
 The above remarks show that Q is indefinite if $a \in (-1, 0) \cup (0, 1)$. If $a = 0$, the quadratic form is $Q(x, y, z) = 4xy + 4yz$ and we see that $Q(1, 1, 0) = 4 > 0$, $Q(1, -1, 0) = -4 < 0$, so Q is indefinite.
 To study the cases $a = \pm 1$ we do the following change of variables

$$\bar{x} = z, \quad \bar{y} = y, \quad \bar{z} = x$$

and we obtain the quadratic form

$$Q(\bar{x}, \bar{y}, \bar{z}) = a\bar{z}^2 + 4a\bar{y}^2 + 4a\bar{x}^2 + 4\bar{z}\bar{y} + 2a\bar{z}\bar{x} + 4\bar{y}\bar{x} = 4a\bar{x}^2 + 4a\bar{y}^2 + a\bar{z}^2 + 4\bar{x}\bar{y} + 2a\bar{z}\bar{x} + 4\bar{y}\bar{x}$$

whose associated matrix is

$$\begin{pmatrix} 4a & 2 & a \\ 2 & 4a & 2 \\ a & 2 & a \end{pmatrix}$$

For this matrix we see that that

$$D_1 = 4a, D_2 = 16a^2 - 4, \quad D_3 = 12a(a^2 - 1)$$

And, for $a = 1$ we obtain that

$$D_1 = 4, D_2 = 8, \quad D_3 = 0$$

so Q is positive semidefinite. Finally, for $a = -1$ we obtain that

$$D_1 = -4, D_2 = 8, \quad D_3 = 0$$

so Q is negative semidefinite.

4-14. *Classify the following quadratic forms, depending on the parameters.*

a) $Q(x,y,z) = 9x^2 + 3y^2 + z^2 + 2axz$

b) $Q(x_1,x_2,x_3) = x_1^2 + 4x_2^2 + bx_3^2 + 2ax_1x_2 + 2x_2x_3$

Solution: a) The matrix associated to $Q(x,y,z) = 9x^2+3y^2+z^2+2axz$ is $\begin{pmatrix} 9 & 0 & a \\ 0 & 3 & 0 \\ a & 0 & 1 \end{pmatrix}$. The principal minors are $D_1 = 9$, $D_2 = \begin{vmatrix} 9 & 0 \\ 0 & 3 \end{vmatrix} = 27$ y $D_3 = \begin{vmatrix} 9 & 0 & a \\ 0 & 3 & 0 \\ a & 0 & 1 \end{vmatrix} = 27 - 3a^2$. Therefore, the quadratic form is

(a) definite positive if $27 - 3a^2 > 0$ that is if, $-3 < a < 3$.
(b) cannot be negative definite since $D_1 = 9 > 0$.
(c) cannot be negative semidefinite either.
(d) is positive semidefinite if $27 - 3a^2 = 0$. That is, if $a = -3$ $a = 3$.
(e) is indefinite if $27 - 3a^2 < 0$. That is, if $|a| > 3$.

b) The matrix associated to $Q(x_1,x_2,x_3) = x_1^2 + 4x_2^2 + bx_3^2 + 2ax_1x_2 + 2x_2x_3$ is $\begin{pmatrix} 1 & a & 0 \\ a & 4 & 1 \\ 0 & 1 & b \end{pmatrix}$. The principal minors are $D_1 = 1 > 0$, $D_2 = \begin{vmatrix} 1 & a \\ a & 4 \end{vmatrix} = 4 - a^2$ y $D_3 = \begin{vmatrix} 1 & a & 0 \\ a & 4 & 1 \\ 0 & 1 & b \end{vmatrix} = 4b - 1 - a^2b = b(4-a^2) - 1$.

Hence,

(a) the quadratic form is positive definite if

$$\left.\begin{array}{r} 4 - a^2 > 0 \\ 4b - 1 - a^2b > 0 \end{array}\right\}$$

From the first inequality we obtain the condition $-2 < a < 2$. De la segunda $b > \frac{1}{4-a^2}$. That is, if

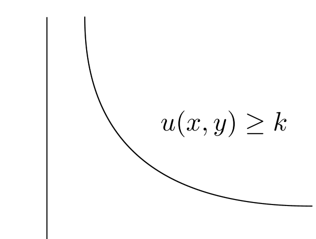
$$\left.\begin{array}{r} -2 < a < 2 \\ b > \frac{1}{4-a^2} \end{array}\right\}$$

(b) the quadratic form cannot be negative definite or semidefinite because $D_1 = 1 > 0$
(c) If $a \in (-2,2)$ y $b = \frac{1}{4-a^2}$, then $D_3 = 4b - 1 - a^2b = 0$ so the quadratic form is positive semidefinite.
(d) If $|a| > 2$ (so, $4 - a^2 < 0$), then the quadratic form is indefinite.
(e) Finally, if $|a| = 2$, we get that $\begin{pmatrix} 1 & a & 0 \\ a & 4 & 1 \\ 0 & 1 & b \end{pmatrix}$. The principal minors are

$$D_1 = 1, \quad D_2 = 4 - a^2 = 0, \quad D_3 = 4b - 1 - a^2b = -1$$

and the quadratic form is indefinite.

4-15. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a concave function so that for every $v_1, v_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we have that $u(\lambda v_1 + (1-\lambda)v_2) \geq \lambda u(v_1) + (1-\lambda)u(v_2)$. Show that $S = \{v \in \mathbb{R}^n : \ u(v) \geq k\}$ is a convex set. For a concave $u : \mathbb{R}^2 \to \mathbb{R}$, the figure represents its graph $S = \{(x,y) \in \mathbb{R}^2 : \ u(x,y) \geq k\}$*



Solution: Let $S = \{x \in \mathbb{R}^n : u(x) \geq k\}$. Let $x, y \in S$, so $u(x) \geq k$ and also $u(y) \geq k$. Given a convex combination of these two points, $x_c = \lambda x + (1-\lambda)y$ we have that

$$\begin{aligned} u(x_c) =& u(\lambda x + (1-\lambda)y) \\ &\geq \lambda u(x) + (1-\lambda)u(y) \qquad \text{since } u \text{ is concave} \\ &\geq \lambda k + (1-\lambda)k = k \end{aligned}$$

Therefore, $x_c \in S$ and S is convex.

4-16. *State the previous problem for a convex function $u : \mathbb{R}^n \to \mathbb{R}$.*

Solution: Let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then, the set $S = \{x \in \mathbb{R}^n : \ u(x) \leq k\}$ is convex.

4-17. *Determine the domains of the plane where the following functions are convex or concave.*
(a) $f(x, y) = (x-1)^2 + xy^2$.
(b) $g(x, y) = \frac{x^3}{3} - 4xy + 12x + y^2$.
(c) $h(x, y) = e^{-x} + e^{-y}$.
(d) $k(x, y) = e^{xy}$.
(e) $l(x, y) = \ln \sqrt{xy}$.

Solution:
(a) First, note that if $x = 0$ then $f(0, y) = 1$ is constant. Hence, f is concave and convex in the set $\{(0, y) : y \in \mathbb{R}\}$. The Hessian matrix of $f(x, y) = (x-1)^2 + xy^2$ is

$$\begin{pmatrix} 2 & 2y \\ 2y & 2x \end{pmatrix}$$

We see that $D_1 = 2 > 0$, $D_2 = 4(x - y^2)$. Since, $D_1 > 0$ the function is not concave in any non-empty subset of $\mathbb{R}^2$. We see that $D_2 \geq 0$ if and only if $x \geq y^2$. The function is convex in the set $\{(x, y) \in \mathbb{R}^2 : x \geq y^2\}$.
(b) The Hessian matrix of

$$f(x, y) = \frac{x^3}{3} - 4xy + 12x + y^2$$

is

$$\begin{pmatrix} 2x & -4 \\ -4 & 2 \end{pmatrix}$$

We see that $D_1 = 2x$, $D_2 = 4x - 16$. The function is concave in the convex sets in which $D_1 < 0$ (so $x < 0$) and $D_2 \geq 0$ (that is, $x \geq 4$). Since, both conditions are not compatible, the function is not concave in any non-empty set of $\mathbb{R}^2$.
If $x > 0$ y $x \geq 4$ then $D_1 > 0$ y $D_2 \geq 0$ and we see that the function is convex in the set $\{(x, y) \in \mathbb{R}^2 : x \geq 4\}$.
(c) The Hessian matrix of $h(x, y) = e^{-x} + e^{-y}$ is

$$\begin{pmatrix} e^{-x} & 0 \\ 0 & e^{-y} \end{pmatrix}$$

Both second derivatives are positive. Hence, the function is convex in $\mathbb{R}^2$.
(d) The Hessian matrix of $k(x, y) = e^{xy}$ is

$$e^{yx} \begin{pmatrix} y^2 & xy+1 \\ xy+1 & x^2 \end{pmatrix}$$

Since, $e^{yx} > 0$ for every $(x, y) \in \mathbb{R}^2$, the signature of the above matrix is the same as the signature of the following one

$$\begin{pmatrix} y^2 & xy+1 \\ xy+1 & x^2 \end{pmatrix}$$

For this matrix we obtain that $D_1 = y^2 \geq 0$, $D_2 = -1 - 2xy$. The function is convex if $D_2 > 0$. That is, if $2xy < -1$. Therefore, the function is convex in the set

$$A = \{(x, y) \in \mathbb{R}^2 : xy < -1/2, x > 0\}$$

and also in the set

$$B = \{(x, y) \in \mathbb{R}^2 : xy < -1/2, x < 0\}$$

The union $A \cup B$ is not a convex set. Finally, in the convex sets $C = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ and $D = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ the function is constant and hence, both concave and convex.
(e) The Hessian matrix of

$$l(x, y) = \ln(\sqrt{xy}) = \begin{cases} \frac{1}{2}(\ln x + \ln y), & \text{if } x, y > 0; \\ \frac{1}{2}(\ln(-x) + \ln(-y)), & \text{if } x, y < 0; \end{cases}$$

is

$$\frac{1}{2} \begin{pmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{pmatrix}$$

Clearly, this matrix is negative definite and, therefore, function is concave in $\mathbb{R}^2_{++}$ and in $\mathbb{R}^2_{--}$.

4-18. *Determine the values of the parameters a and b so that the following functions are convex in their domains.*
(a) $f(x,y,z) = ax^2 + y^2 + 2z^2 - 4axy + 2yz$
(b) $g(x,y) = 4ax^2 + 8xy + by^2$

Solution:
(a) The Hessian of $f(x,y,z) = ax^2 + y^2 + 2z^2 - 4axy + 2yz$ is

$$\begin{pmatrix} 2a & -4a & 0 \\ -4a & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

Note that

$$\begin{aligned} D_1 &= 2a \\ D_2 &= \begin{vmatrix} 2a & -4a \\ -4a & 2 \end{vmatrix} = 4a - 16a^2 = 4a(1-4a) \\ D_3 &= \begin{vmatrix} 2a & -4a & 0 \\ -4a & 2 & 2 \\ 0 & 2 & 4 \end{vmatrix} = 8a - 64a^2 = 8a(1-8a) \end{aligned}$$

Thus, $D_1 > 0$ is equivalent to $a > 0$. Assuming this, the condition $D_3 > 0$ is equivalent to $a < 1/8$. Furthermore, if $0 < a < 1/8$ then $D_2 > 0$, so the function is strictly convex if $0 < a < 1/8$. On the other hand, if $a = 0$ or $a = 1/8$, the Hessian positive semidefinite. Therefore, the function is convex if $0 \le a \le 1/8$.
(b) The Hessian of $g(x,y) = 4ax^2 + 8xy + by^2$ is

$$\begin{pmatrix} 8a & 8 \\ 8 & 2b \end{pmatrix}$$

Note that

$$\begin{aligned} D_1 &= 8a \\ D_2 &= \begin{vmatrix} 8a & 8 \\ 8 & 2b \end{vmatrix} = 16(ab - 4) \end{aligned}$$

The function is convex if $a > 0$ and $ab \ge 4$. This is equivalent to $a > 0$ and $b \ge 4/a$.
If $a = 0$, then $D_1 = 0$, $D_2 = -64 \neq 0$. Hence, $\mathrm{H}\,h(x,y)$ is indefinite for every $(x,y) \in \mathbb{R}^2$ and the function is not convex in $\mathbb{R}^2$.
If $a < 0$, then $D_1 < 0$, so $\mathrm{H}\,h(x,y)$ cannot be positive definite or positive semidefinite at any $(x,y) \in \mathbb{R}^2$ and the function is not convex in $\mathbb{R}^2$.

4-19. *Discuss the concavity and convexity of the function $f(x,y) = -6x^2 + (2a+4)xy - y^2 + 4ay$ according to the values of a.*

Solution: The Hessian of $f(x,y) = -6x^2 + (2a+4)xy - y^2 + 4ay$ is

$$\begin{pmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{pmatrix}$$

We have that

$$\begin{aligned} D_1 &= -12 < 0 \\ D_2 &= \begin{vmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{vmatrix} = 8 - 4a^2 - 16a \end{aligned}$$

Since $D_1 < 0$ the function cannot be convex. It would be concave if $D_2 = 8 - 4a^2 - 16a \ge 0$. The roots of $8 - 4a^2 - 16a = 0$ are $-2 \pm \sqrt{6}$. Thus, $D_2 \ge 0$ is equivalent to $-2 - \sqrt{6} \le a \le -2 + \sqrt{6}$. Therefore f is concave if $a \in [-2-\sqrt{6}, -2+\sqrt{6}]$.

4-20. *Find the largest convex set of the plane where the function $f(x,y) = x^2 - y^2 - xy - x^3$ is concave.*

Solution: The Hessian of $f(x,y) = x^2 - y^2 - xy - x^3$ is

$$\begin{pmatrix} 2-6x & -1 \\ -1 & -2 \end{pmatrix}$$

We have that

$$D_1 = 2 - 6x$$
$$D_2 = 12x - 5$$

The condition $D_2 \geq 0$ is equivalent to $x \geq 5/12$. Since $5/12 > 1/3$, the previous inequality also guarantees that $D_1 < 0$. Therefore, the largest set of the plane in which f is concave is the set $\{(x, y) \in \mathbb{R}^2 : x \geq 5/12\}$.