CHAPTER 2: Limits and continuity of functions in $\mathbb{R}^n$.

2-1. Sketch the following subsets of $\mathbb{R}^2$. Sketch their boundary and the interior. Study whether the following are closed, open, bounded and/or convex.
(a) $A = \{(x, y) \in \mathbb{R}^2 : 0 < \|(x, y) - (1, 3)\| < 2\}$.
(b) $B = \{(x, y) \in \mathbb{R}^2 : y \leq x^3\}$.
(c) $C = \{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| \leq 2\}$.
(d) $D = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$.
(e) $E = \{(x, y) \in \mathbb{R}^2 : y < x^2, y < 1/x, x > 0\}$.
(f) $F = \{(x, y) \in \mathbb{R}^2 : xy \leq y + 1\}$.
(g) $G = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 \leq 1, x \leq 1\}$.

2-2. Find the domain of the following functions.
(a) $f(x, y) = (x^2 + y^2 - 1)^{1/2}$.
(b) $f(x, y) = \dfrac{1}{xy}$.
(c) $f(x, y) = e^x - e^y$.
(d) $f(x, y) = e^{xy}$.
(e) $f(x, y) = \ln(x + y)$.
(f) $f(x, y) = \ln(x^2 + y^2)$.
(g) $f(x, y, z) = \sqrt{\frac{x^2+1}{yz}}$.
(h) $f(x, y) = \sqrt{x - 2y + 1}$.

2-3. Find the range of the following functions.
(a) $f(x, y) = (x^2 + y^2 + 1)^{1/2}$.
(b) $f(x, y) = \dfrac{xy}{x^2 + y^2}$.
(c) $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$.
(d) $f(x, y) = \ln(x^2 + y^2)$.
(e) $f(x, y) = \ln(1 + x^2 + y^2)$.
(f) $f(x, y) = \sqrt{x^2 + y^2}$.

2-4. Draw the level curves of the following functions.
(a) $f(x, y) = xy$, $c = 1, -1, 3$.
(b) $f(x, y) = e^{xy}$, $c = 1, -1, 3$.
(c) $f(x, y) = \ln(xy)$, $c = 0, 1, -1$.
(d) $f(x, y) = (x + y)/(x - y)$, $c = 0, 2, -2$.
(e) $f(x, y) = x^2 - y$, $c = 0, 1, -1$.
(f) $f(x, y) = ye^x$, $c = 0, 1, -1$.

2-5. Let $f(x, y) = Cx^{\alpha}y^{1-\alpha}$, with $0 < \alpha < 1$ and $C > 0$ be the Cobb-Douglas production function, where x (resp. y) represents units of labor (resp. capital) and f are the units produced.
(a) Represent the level curves of f.
(b) Show that if one duplicates labor and capital then, production is doubled, as well.

2-6. Study the existence and the value of the following limits.
(a) $\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$.
(b) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$.
(c) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^4+y^2}$.

(d) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+2y^2}$.
(e) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.
(f) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$.
(g) $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2}$.

2-7. Study the continuity of the following functions.

(a) $f(x,y) = \begin{cases} \frac{x^2y}{x^3+y^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

(b) $f(x,y) = \begin{cases} \frac{xy+1}{y}x^2 & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$.

(c) $f(x,y) = \begin{cases} \frac{x^4y}{x^6+y^3} & \text{if } y \neq -x^2 \\ 0 & \text{if } y = -x^2 \end{cases}$.

(d) $f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{si } (x,y) = (0,0) \end{cases}$.

2-8. Consider the set $A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x, \quad \leq 1, \quad 0 \leq y \leq 1\}$ and the function $f\colon A \longrightarrow \mathbb{R}^2$, defined by

$$f(x,y) = \left(\frac{x+1}{y+2}, \frac{y+1}{x+2}\right)$$

Are the hypotheses of Brouwer's Theorem satisfied? Is it possible to determine the fixed point(s)?

2-9. Consider the function $f(x,y) = 3y - x^2$ defined on the set $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, \quad 0 \leq x < 1/2, \quad y \geq 0\}$. Draw the set D and the level curves of f. Does f have a maximum and a minimum on D?

2-10. Consider the sets $A = \{(x,y) \in \mathbb{R}^2 | 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and $B = \{(x,y) \in \mathbb{R}^2 | -1 \leq x \leq 1, -1 \leq y \leq 1\}$ and the function

$$f(x,y) = \frac{(x+1)\left(y+\frac{1}{5}\right)}{y+\dfrac{1}{2}}$$

What can you say about the extreme points of f on A and B?

2-11. Consider the set

$$A = \{(x,y) \in \mathbb{R}^2\colon 0 \leq y \leq \ln x, 1 \leq x \leq 2\}.$$

(a) Draw the set A, its boundary and its interior. Discuss whether the set A is open, closed, bounded, compact and/or convex. You must explain your answer.
(b) Prove that the function $f(x,y) = y^2 + (x-1)^2$ has a maximum and a minimum on A.
(c) Using the level curves of $f(x,y)$, find the maximum and the minimum of f on A.

2-12. Consider the set $A = \{(x,y) \in \mathbb{R}^2 : x, y > 0; \ln(xy) \geq 0\}$.
(a) Draw the set A, its boundary and its interior. Discuss whether the set A is open, closed, bounded, compact and/or convex. You must explain your answer.
(b) Consider the function $f(x,y) = x + 2y$. Is it possible to use Weierstrass' Theorem to determine whether the function attains a maximum and a minimum on A? Draw the level curves of f, indicating the direction in which the function grows.
(c) Using the level curves of f, find graphically (i.e. without using the first order conditions) if f attains a maximum and/or a minimum on A.