UNIVERSIDAD CARLOS III DE MADRID

MATHEMATICS FOR ECONOMICS II | SPRING 2024

EXERCISES (SOLUTIONS)

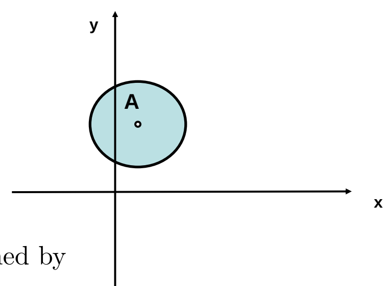
CHAPTER 2: Limits and continuity of functions in $\mathbb{R}^n$.

2-1. *Sketch the following subsets of $\mathbb{R}^2$. Sketch their boundary and the interior. Study whether the following are closed, open, bounded, compact and/or convex.*

(a) $A = \{(x, y) \in \mathbb{R}^2 : 0 < \|(x, y) - (1, 3)\| < 2\}$.
(b) $B = \{(x, y) \in \mathbb{R}^2 : y \leq x^3\}$.
(c) $C = \{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| \leq 2\}$.
(d) $D = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$.
(e) $E = \{(x, y) \in \mathbb{R}^2 : y < x^2, y < 1/x, x > 0\}$.
(f) $F = \{(x, y) \in \mathbb{R}^2 : xy \leq y + 1\}$.
(g) $G = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 \leq 1, x \leq 1\}$.

Solution:

(a) The set represents the disk of center $C = (1, 3)$ and radius 2 with the center removed.



The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \|(x, y) - (1, 3)\| = \sqrt{(x-1)^2 + (y-3)^2}$$

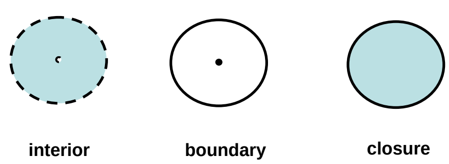
is continuous and the set A may be written as

$$A = \{(x, y) \in \mathbb{R}^2 : 0 < f(x, y) < 2\} = \{(x, y) \in \mathbb{R}^2 : f(x, y) \in (0, 2)\}$$

Since, the interval $(0, 2) \subset \mathbb{R}$ is open, the set A is **open**. It is also **bounded**, since it is contained in the disc $\{(x, y) \in \mathbb{R}^2 : \|(x, y) - (1, 3)\| < 2\}$.
In addition, **it is not convex** since the points $P = (1, 4)$ and $Q = (1, 2)$ belong to A but the convex combination

$$\frac{1}{2}(1, 4) + \frac{1}{2}(1, 2) = (1, 3)$$

does not belong to the set A.
The interior, boundary and closure of A are represented in the following figure



Note that $\partial A \cap A = \emptyset$. This gives another way to prove that the set A is open.

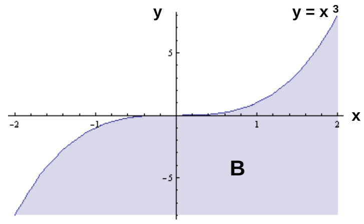
(b) The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^3 - y$$

is continuous and the set B may be written as

$$B = \{(x, y) \in \mathbb{R}^2 : f(x, y) \geq 0\} = \{(x, y) \in \mathbb{R}^2 : f(x, y) \in [0, \infty)\}$$

Since, the interval $[0, \infty) \subset \mathbb{R}$ is closed, the set B is **closed**.



The set B **is not bounded** since, for example, the points

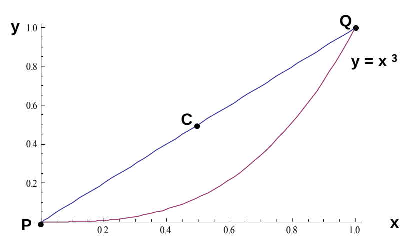
$$(1,0),(2,0),\dots,(n,0),\dots$$

belong to B and

$$\lim_{n\to\infty}\|(n,0)\|=+\infty$$

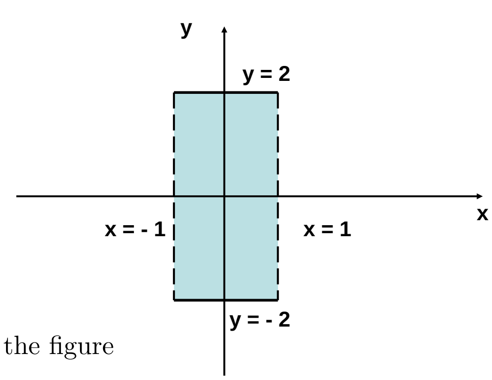
Furthermore, **it is not convex** since the points $P=(0,0)$ and $Q=(1,1)$ belong to B but the convex combination

$$C=\frac{1}{2}P+\frac{1}{2}Q=\left(\frac{1}{2},\frac{1}{2}\right)$$

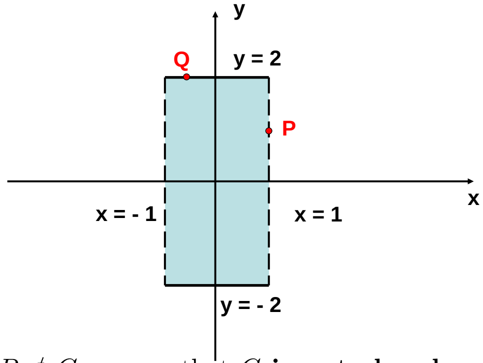


does not belong to B, because it does not satisfy the equation $y\le x^3$.
The interior of B is the set $\{(x,y)\in\mathbb{R}^2 : y<x^3\}$. The boundary of B is the set $\partial(B)=\{(x,y)\in\mathbb{R}^2 : y=x^3\}$. And the closure of B is the set $\bar{B}=B\cup\partial(B)=\{(x,y)\in\mathbb{R}^2 : y\le x^3\}$. Since, $\bar{B}=B$, the set **is closed**.

(c) Graphically, the set C is



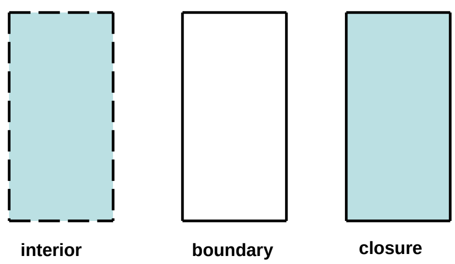
The points P and Q in the figure



belong to $\partial(C)$. Since, $P \notin C$, we see that C **is not closed** and since $Q \in C$, we see that C **is not open**.
Graphically, we see that the set C **is convex**. An alternative way to prove this is by noting that the set C is determined by the following **linear inequalities**

$$x > -1, \qquad x < 1, \qquad y \geq -2, \qquad y \leq 2$$

The interior, boundary and closure of A are represented in the following figure

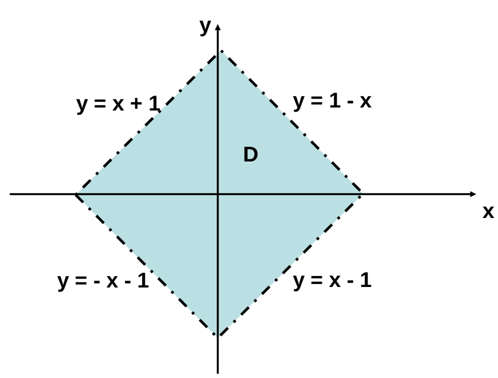


We see that $\partial(C) \cap C \neq \emptyset$, so the set **is not open**. Furthermore, $C \neq \bar{C}$ so the set **is not closed**.

(d) The following functions defined from $\mathbb{R}^2$ into $\mathbb{R}$ are continuous.

$$\begin{aligned} f_1(x,y) &= y - x - 1 \\ f_2(x,y) &= y - 1 + x \\ f_3(x,y) &= y + x + 1 \\ f_4(x,y) &= y - x + 1 \end{aligned}$$

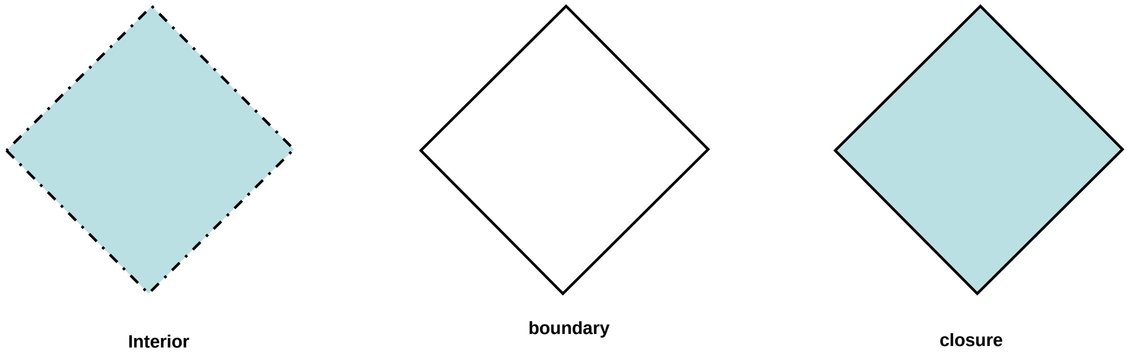
The set D



is defined by

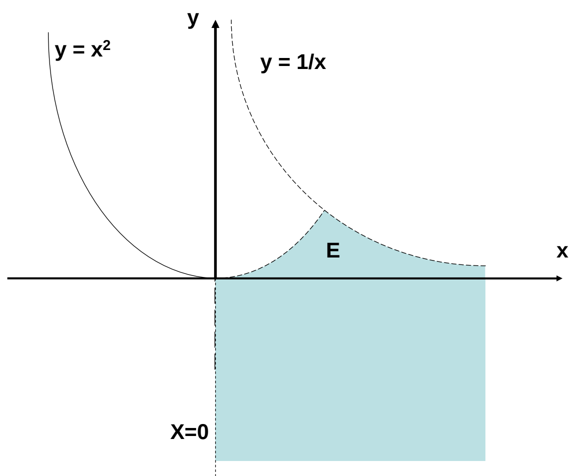
$$D = \{(x,y) \in \mathbb{R}^2 : f_1(x,y) < 0, \quad f_2(x,y) < 0, \quad f_3(x,y) > 0, \quad f_4(x,y) > 0\}$$

so it **is open and convex**. The set D **is bounded** because is contained in the disc of center $(0,0)$ and radius 1.
The interior, boundary and closure of A are represented in the following figure



Since, $\partial(D) \cap D = \emptyset$, the set **is open**.

(e) The graphic representation of E is



The functions

$$\begin{aligned} f_1(x,y) &= y - x^2 \\ f_2(x,y) &= y - 1/x \\ f_3(x,y) &= x \end{aligned}$$

are defined from $\mathbb{R}^2$ into $\mathbb{R}$ and are continuous. The set E is defined by

$$E = \{(x,y) \in \mathbb{R}^2 : f_1(x,y) < 0, f_2(x,y) < 0, f_3(x,y) > 0\}$$

so it **is open**. The set E **is not bounded** because the points

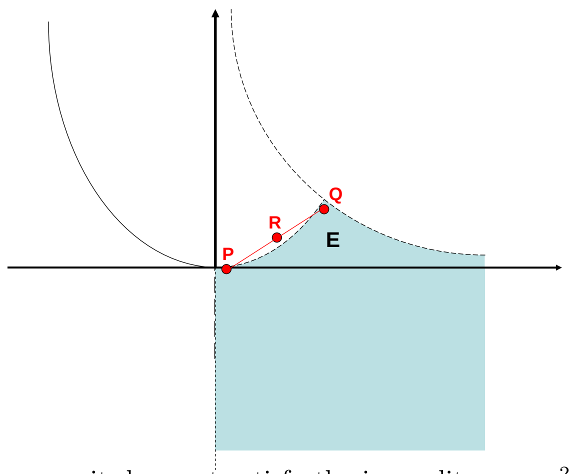
$$(n,0) \quad n = 1, 2, \dots$$

belong to E and

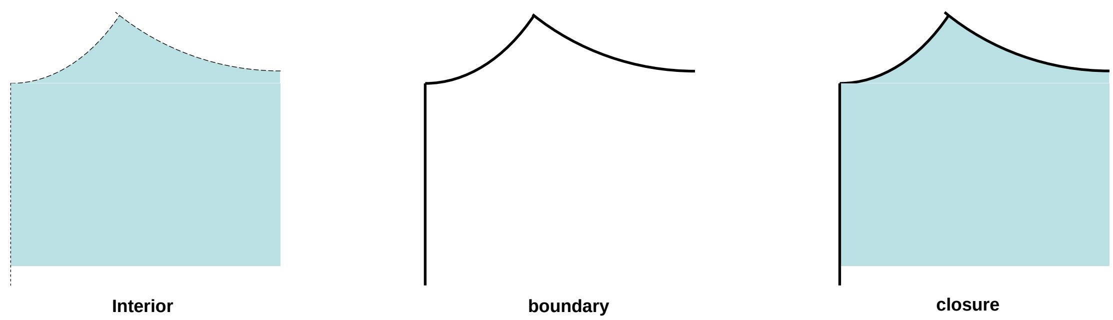
$$\lim_{n\to\infty} \| (n,0) \| = \lim_{n\to\infty} n = +\infty$$

In addition, **it is not convex** because the points $P = (0'2, 0)$ and $Q = (1, 0'8)$ belong to E but the convex combination

$$R = \frac{1}{2}P + \frac{1}{2}Q = (0'6, 0'4)$$

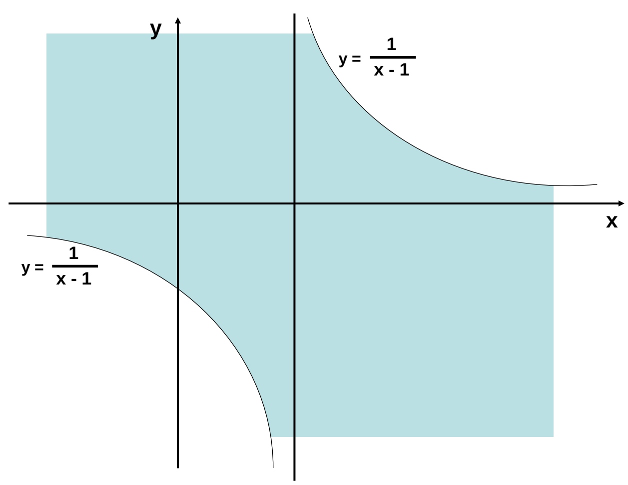


does not belong to E, because it does not satisfy the inequality $y < x^2$. The interior, boundary and closure of E are represented in the following figure



Since $\partial(E) \cap E = \emptyset$, the set **is open**.

(f) Graphically, the F is



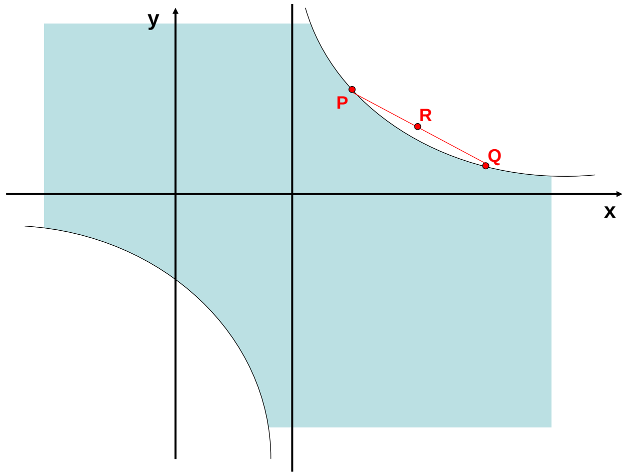
The function $f(x, y) = xy - y$ defined from $\mathbb{R}^2$ into $\mathbb{R}$ is continuous. The set F is $F = \{(x, y) \in \mathbb{R}^2 : f(x, y) \leq 1\}$ so **is closed**. The set F **is not bounded** because the points

$$(n, 0) \quad n = 1, 2, \ldots$$

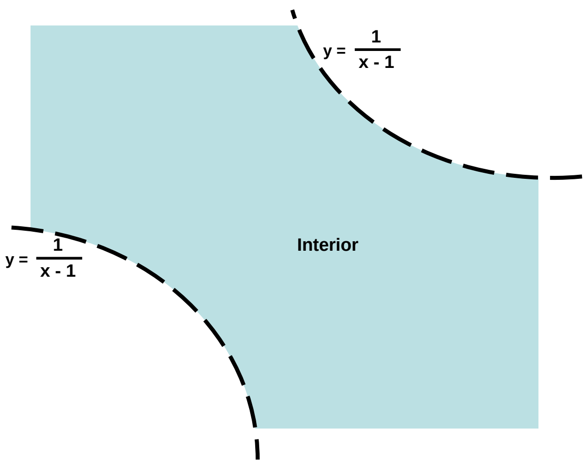
are in E and

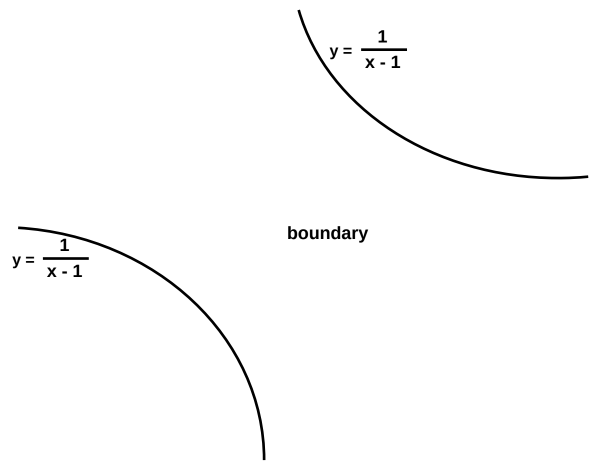
$$\lim_{n\to\infty} \| (n, 0) \| = \lim_{n\to\infty} n = +\infty$$

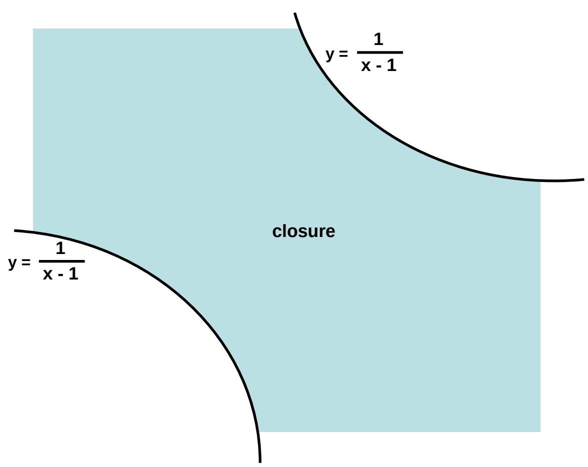
The figure



shows why F **is not convex**. The interior, closure and boundary of F are represented in the following figure

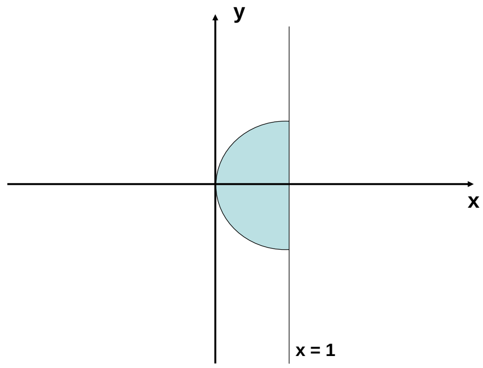




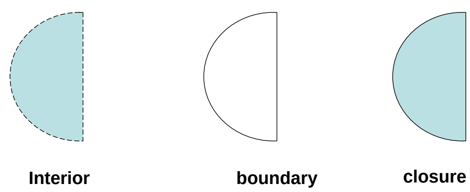


Since $\partial(F) \subset F$, the set F **is closed**.

(g) Graphically the set G is



The functions $f(x, y) = (x-1)^2 + y^2$ and $g(x, y) = x$ defined from $\mathbb{R}^2$ into $\mathbb{R}$ are continuous. The set G is $G = \{(x, y) \in \mathbb{R}^2 : f(x, y) \leq 1, \quad g(x, y) \leq 1\}$ so it **is closed**. The set G **is bounded** because it is contained in the disc of center $(1, 0)$ and radius 1. Further, the set G **is convex**. The interior, boundary and the closure of G are represented in the following figure



Since $\partial(G) \subset G$, the set G is **closed**.

2-2. *Find the domain of the following functions.*

(a) $f(x, y) = (x^2 + y^2 - 1)^{1/2}$.

(b) $f(x, y) = \dfrac{1}{xy}$.

(c) $f(x, y) = e^x - e^y$.

(d) $f(x, y) = e^{xy}$.

(e) $f(x, y) = \ln(x + y)$.

(f) $f(x, y) = \ln(x^2 + y^2)$.

(g) $f(x, y, z) = \sqrt{\frac{x^2+1}{yz}}$.

(h) $f(x, y) = \sqrt{x - 2y + 1}$.

Solution:

(a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$.

(b) $\{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$. ($\mathbb{R}^2$ except the axes).

(c) $\mathbb{R}^2$.

(d) $\mathbb{R}^2$.

(e) $\{(x, y) \in \mathbb{R}^2 : x + y > 0\}$.

(f) $\mathbb{R}^2 \setminus \{(0, 0)\}$.

(g) $\{(x, y, z) \in \mathbb{R}^3 : yz > 0\}$.

(h) $\{(x, y) \in \mathbb{R}^2 : x - 2y \geq -1\}$.

2-3. *Find the range of the following functions.*

(a) $f(x, y) = (x^2 + y^2 + 1)^{1/2}$.

(b) $f(x, y) = \dfrac{xy}{x^2 + y^2}$.

(c) $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$.

(d) $f(x,y) = \ln(x^2 + y^2)$.
(e) $f(x,y) = \ln(1 + x^2 + y^2)$.
(f) $f(x,y) = \sqrt{x^2 + y^2}$.

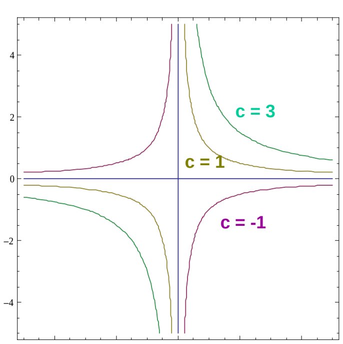
Solution:
(a) $[1, \infty)$.
(b) $\left[\frac{-1}{2}, \frac{1}{2}\right]$.
(c) $[-1, 1]$.
(d) $(-\infty, \infty)$.
(e) $[0, \infty)$.
(f) $[0, \infty)$.

2-4. *Draw the level curves of the following functions.*
(a) $f(x,y) = xy$, $c = 1, -1, 3$.
(b) $f(x,y) = e^{xy}$, $c = 1, -1, 3$.
(c) $f(x,y) = \ln(xy)$, $c = 0, 1, -1$.
(d) $f(x,y) = (x+y)/(x-y)$, $c = 0, 2, -2$.
(e) $f(x,y) = x^2 - y$, $c = 0, 1, -1$.
(f) $f(x,y) = ye^x$, $c = 0, 1, -1$.

Solution:
(a) The level curves are determined by the equation $xy = c$. For $c \neq 0$, this equation is equivalent

$$y = \frac{c}{x}$$

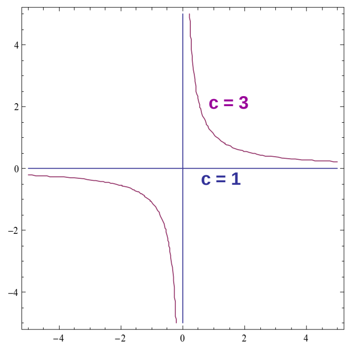
so the level curves are



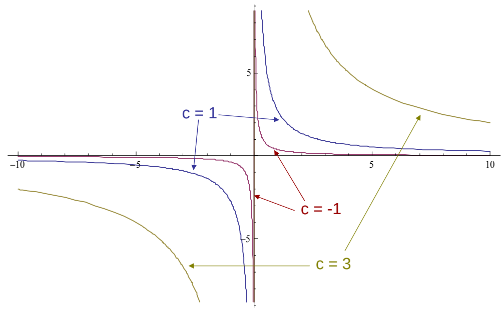
(b) The level curves are determined by the equation $e^{xy} = c$. Therefore, the level curve corresponding to $c \leq 0$ is the empty set. In particular, there is no level curve corresponding to $c = -1$.
For $c > 0$, the level curve satisfies the equation $e^{xy} = c$, so $xy = \ln c$. For $c = 1$, the level curve consists of the points $(x,y) \in \mathbb{R}^2$ such that $xy = 0$. For $c = 3$, the level curve consists of the points $(x,y) \in \mathbb{R}^2$ such that

$$y = \frac{\ln 3}{x}$$

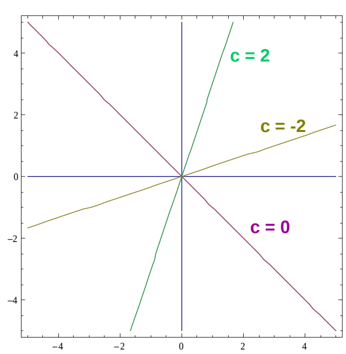
Graphically,



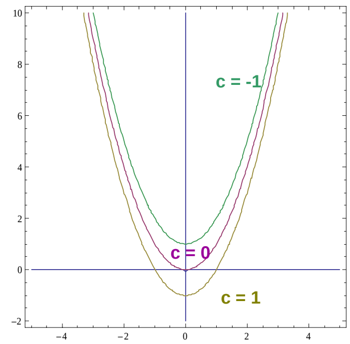
(c) The level curves are determined by the equation $\log(xy) = c$ which is the same as $xy = e^c$. Graphically,



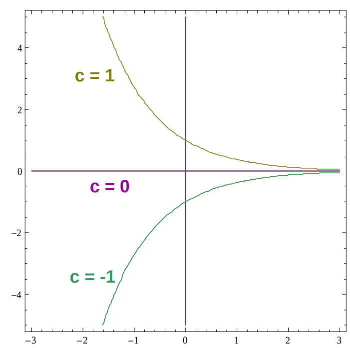
(d) The level curves are determined by the equation $x+y = c(x-y)$, which is the same as $(1+c)y = (c-1)x$. Graphically,



(e) The level curves satisfy the equation $y = x^2 - c$. Graphically,

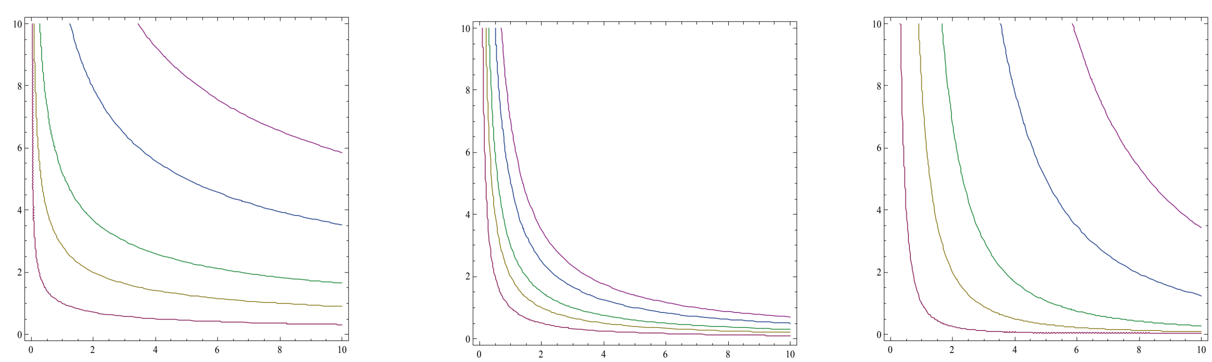


(f) The level curves satisfy the equation $y = ce^{-x}$. Graphically,



2-5. *Let $f(x, y) = Cx^{\alpha}y^{1-\alpha}$, with $0 < \alpha < 1$ and $C > 0$ be the Cobb-Douglas production function, where x (resp. y) represents units of labor (resp. capital) and f are the units produced.*
(a) *Represent the level curves of f.*
(b) *Show that if one duplicates labor and capital then, production is doubled, as well.*

Solution:
(a) The level curves are,



$\alpha < 1/2$ $\alpha = 1/2$ $\alpha > 1/2$

(b) $f(x, y) = Cx^{\alpha}y^{1-\alpha}$, $f(2x, 2y) = C\,(2x)^{\alpha}\,(2y)^{1-\alpha} = 2Cx^{\alpha}y^{1-\alpha} = 2f(x, y)$.

2-6. *Study the existence and the value of the following limits.*
(a) $\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$.
(b) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$.

(c) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^4+y^2}$.
(d) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+2y^2}$.
(e) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.
(f) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$.
(g) $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2}$.

Solution:

(a) $\left[(\frac{x}{x^2+y^2})\right]_{y=x} = \frac{x}{x^2+x^2} = \frac{1}{2x}$ and $\lim_{x\to 0}\left(\frac{1}{2x}\right)$ does not exist. The limit does not exist.

(b) We show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Note that

$$0 \le |f(x,y)| = \left|\frac{xy^2}{x^2+y^2}\right| \le \frac{|x|(x^2+y^2)}{x^2+y^2} = |x|$$

and $\lim_{(x,y)\to(0,0)} |x| = 0$. Hence, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(c) On the one hand,

$$\left[(\frac{3x^2y}{x^4+y^2})\right]_{y=x} = \frac{3x^3}{x^4+x^2} = \frac{3x}{x^2+1}$$

and

$$\lim_{x\to 0} \frac{3x}{x^2+1} = 0$$

On the other hand,

$$\left[(\frac{3x^2y}{x^4+y^2})\right]_{y=x^2} = \frac{3}{2}$$

Therefore, the limit does not exist.

(d) $\left[(\frac{x^2-y^2}{x^2+2y^2})\right]_{y=kx} = \frac{x^2-k^2x^2}{x^2+2k^2x^2} = \frac{1-k^2}{1+2k^2}$, depends on k. Therefore, the limit does not exist.

(e) $\left[(\frac{xy}{x^2+y^2})\right]_{y=kx} = x^2\frac{k}{x^2+k^2x^2} = \frac{k}{1+k^2}$, depends on k.

(f) We show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Note that

$$|f(x,y)| = \left|\frac{x^2y}{x^2+y^2}\right| \le \frac{(x^2+y^2)|y|}{x^2+y^2} = |y|$$

and $\lim_{(x,y)\to(0,0)} |y| = 0$. Hence, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(g) We show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Note that

$$|f(x,y)-0| = \left|\frac{xy^3}{x^2+y^2}\right| = \left|\frac{y^2}{x^2+y^2}xy\right| \le \frac{x^2+y^2}{x^2+y^2}|xy| = |xy|$$

and $\lim_{(x,y)\to(0,0)} |xy| = 0$. Hence, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

2-7. *Study the continuity of the following functions.*

(a) $f(x,y) = \begin{cases} \frac{x^2y}{x^3+y^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

(b) $f(x,y) = \begin{cases} \frac{xy+1}{y}x^2 & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$.

(c) $f(x,y) = \begin{cases} \frac{x^4y}{x^6+y^3} & \text{if } y \neq -x^2 \\ 0 & \text{if } y = -x^2 \end{cases}$.

(d) $f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{si } (x,y) = (0,0) \end{cases}$.

Solution:

(a) The function $\frac{x^2y}{x^3+y^3}$ is not continuous at the points $\{(x,y) : x = -y\}$. (The limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$$

does not exist. This can be shown by taking curves of the form $y = kx$.)

(b) The function $\frac{xy+1}{y}x^2$,

(i) is continuous at the points (x,y) such that $y \neq 0$.

(ii) is not continuous at the points of the form $(x_0, 0)$ with $x_0 \neq 0$. Since, the limit

$$\lim_{y\to 0}(f(x_0,y)) = \lim_{y\to 0} x_0^3 + \frac{x_0^2}{y}$$

does not exist if $x_0 \neq 0$.

(iii) It is not continuous at $(0,0)$ because

$$\lim_{x\to 0} f(x, kx^2) = \lim_{x\to 0}\left(x^3 + \frac{1}{k}\right) = \frac{1}{k}$$

which depends on k.

(c) The function

$$\frac{x^4y}{x^6+y^3}$$

is continuous at the points (x,y) such that $y \neq -x^2$. On the other hand, at the points of the form $(a, -a^2)$ is not continuous because,

(i) If $a \neq 0$, we have that

$$\lim_{y\to -a^2} f(a,y)$$

does not exist because the numerator approaches $-a^6 \neq 0$ whereas the denominator approaches 0.

(ii) The limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist because

$$\lim_{t\to 0} f(t,t^2) = \lim_{t\to 0} \frac{t^6}{t^6+t^6} = \frac{1}{2}$$

whereas the value of the iterated limits is 0.

(d) The function $\frac{xy^3}{x^2+y^2}$ is a quotient of polynomials and the denominator only vanishes at the point $(x,y) = (0,0)$. Hence, the function is continuous at every point $(x,y) \neq (0,0)$.
At the point $(0,0)$ the function is also continuous because we have already proved in another problem that

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2} = 0$$

We conclude that the function is continuous in all of $\mathbb{R}^2$.

2-8. *Consider the set* $A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x, \quad \leq 1, \quad 0 \leq y \leq 1\}$ *and the function* $f\colon A \longrightarrow \mathbb{R}^2$*, defined by*

$$f(x,y) = \left(\frac{x+1}{y+2}, \frac{y+1}{x+2}\right)$$

Are the hypotheses of Brouwer's Theorem satisfied? Is it possible to determine the fixed point(s)?

Solution: Brouwer's Theorem: Let A be a compact, non-empty and convex subset of R^n and let $f : A \to A$ be a continuous function. Then, f has a unique fixed point. (That is, a point $a \in A$, such that $f(a) = a$). The set A is not empty, compact and convex. The function f is continuous if $y \neq -2$ and $x \neq -2$. Therefore, f is continuous on A and Brouwer's Theorem applies.

If (x,y) is the fixed point of f, then

$$x = \frac{x+1}{y+2}$$
$$y = \frac{y+1}{x+2}$$

that is,

$$xy = 1-x$$
$$xy = 1-y$$

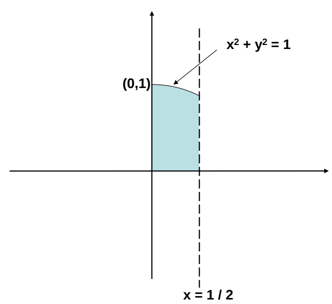
Therefore $x=y$ satisfies the equation $x^2+x-1=0$ whose solutions are

$$x = \frac{-1\pm\sqrt{5}}{2}$$

The only solution in the set A is $(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$.

2-9. *Consider the function $f(x,y) = 3y - x^2$ defined on the set $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1, \quad 0 \le x < 1/2, \quad y \ge 0\}$. Draw the set D and the level curves of f. Does f have a maximum and a minimum on D?*

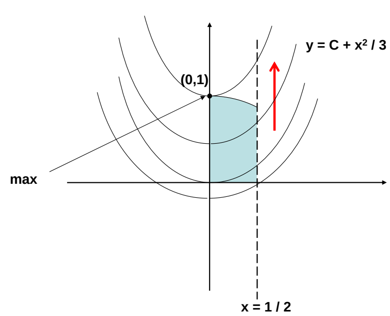
Solution: The set D is the following



Note that D is not compact (since it is not closed). It does not contain the point $(1/2,0)$. On the other hand, the level curves of f are of the form

$$y = C + \frac{x^2}{3}$$

Graphically, (the red arrow points in the direction of growth)



We see that f attains a maximum at the point $(0,1)$, but attains no minimum on A.

2-10. *Consider the sets $A = \{(x,y) \in \mathbb{R}^2 | 0 \le x \le 1, 0 \le y \le 1\}$ and $B = \{(x,y) \in \mathbb{R}^2 | -1 \le x \le 1, -1 \le y \le 1\}$ and the function*

$$f(x,y) = \frac{(x+1)\left(y+\frac{1}{5}\right)}{y+\dfrac{1}{2}}$$

What can you say about the extreme points of f on A and B?

Solution: The function

$$f(x,y) = \frac{(x+1)\left(y+\frac{1}{5}\right)}{y+\frac{1}{2}}$$

is continuous if $y \neq -1/2$ and so, is continuous in the set A, which is compact. By Weierstrass' Theorem, f attains a maximum and a minimum on A.

But, for example, the point $(0,-1/2) \in \mathrm{Int}B$ and

$$\lim_{y\to(\frac{-1}{2})^+} f(0,y) = -\infty, \qquad \lim_{y\to(\frac{-1}{2})^-} f(0,y) = +\infty$$

so f does not attain neither a maximum nor a minimum on B.

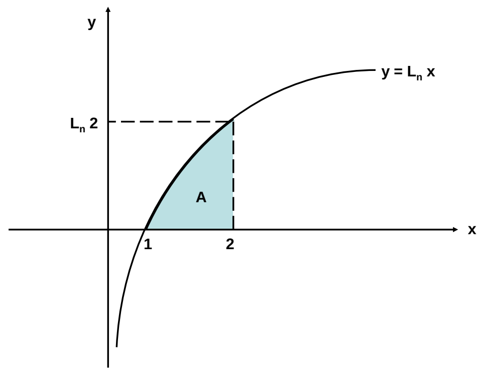
2-11. *Consider the set*

$$A = \{(x,y) \in \mathbb{R}^2 \colon 0 \le y \le \ln x, 1 \le x \le 2\}.$$

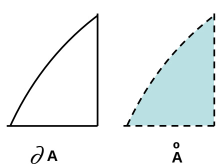
(a) *Draw the set A, its boundary and its interior. Discuss whether the set A is open, closed, bounded, compact and/or convex. You must explain your answer.*
(b) *Prove that the function $f(x,y) = y^2 + (x-1)^2$ has a maximum and a minimum on A.*
(c) *Using the level curves of $f(x,y)$, find the maximum and the minimum of f on A.*

Solution:

(a) The set A is



The boundary and the interior are



Since $\partial A \subset A$, the set A is closed. It is not open because $\partial A \cap A \neq \emptyset$. Another way of proving this, would be to consider the sets $A_1 = \{(x,y) \in \mathbb{R}^2 \colon 0 \le y\}$, $A_2 = \{(x,y) \in \mathbb{R}^2 \colon 1 \le x \le 2\}$. The set $A_3 = \{(x,y) \in \mathbb{R}^2 \colon y \le \log(x)\}$ is also closed since the function $g(x,y) = log(x) - y$ is continuous. Therefore, $A = A_1 \cap A_2 \cap A_3$ is a closed set.

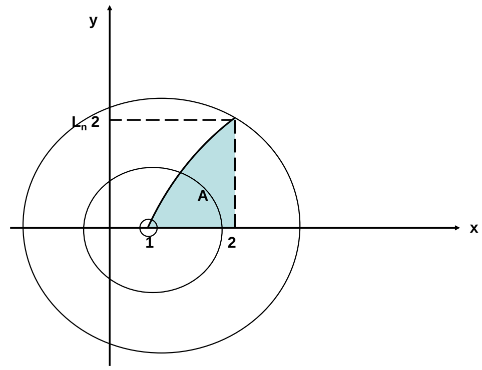
The set A is bounded since $A \subseteq B(0,r)$ with $r > 0$ large enough. Since it is closed and bounded the set A is compact. The set A is convex since is the region under the graph of $f(x) = \ln x$ in the interval $[1,2]$ and the function $\ln x$ is concave.

(b) The function f is continuous in $\mathbb{R}^2$, since it is a polynomial. In particular, the function is continuous in the set A. Furthermore, the set A is compact. By Weierstrass' Theorem, the function attains a maximum and a minimum on A.

(c) The equations defining the level curves of f are

$$f(x,y) = y^2 + (x-1)^2 = C$$

These sets are circles centered at the point $(1,0)$ and radius $\sqrt{C}$, for $C \ge 0$.



Graphically, we see that the maximum is $f(2, \ln 2) = 1 + (\ln 2)^2$ and is attained at the point $(2, \ln 2)$. The minimum is $f(1, 0) = 0$ and is attained at the point $(1, 0)$.

2-12. *Consider the set* $A = \{(x, y) \in \mathbb{R}^2 : x, y > 0; \ln(xy) \geq 0\}$.

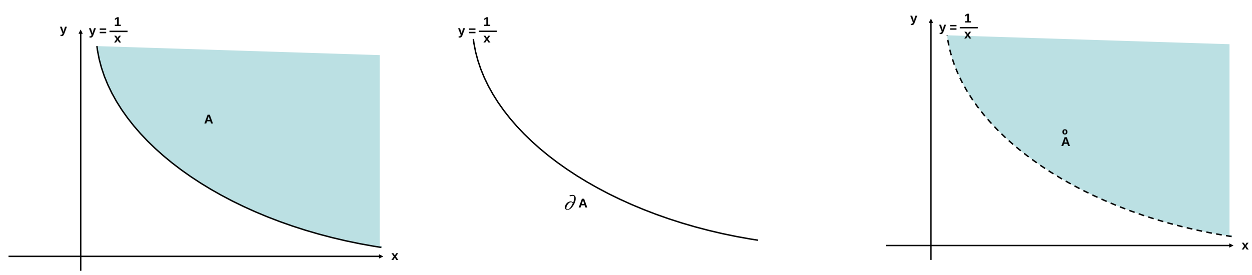
(a) *Draw the set* A*, its boundary and its interior. Discuss whether the set* A *is open, closed, bounded, compact and/or convex. You must explain your answer.*

(b) *Consider the function* $f(x, y) = x + 2y$*. Is it possible to use Weierstrass' Theorem to determine whether the function attains a maximum and a minimum on* A*? Draw the level curves of* f*, indicating the direction in which the function grows.*

(c) *Using the level curves of* f*, find graphically (i.e. without using the first order conditions) if* f *attains a maximum and/or a minimum on* A*.*

Solution:

(a) The equation $\ln(xy) \geq 0$ is equivalent to $xy \geq 1$. Since $x, y > 0$, the set is $A = \{(x, y) \in \mathbb{R}^2 : y \geq 1/x, \quad x > 0\}$. Graphically,



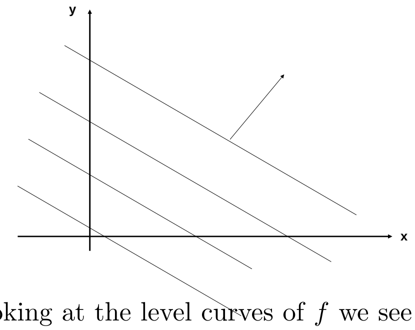
The boundary is the set $A = \{(x, y) \in \mathbb{R}^2 : y = 1/x, \quad x > 0\}$. The interior is the set $\overset{\circ}{A} = \{(x, y) \in \mathbb{R}^2 : y > 1/x, \quad x > 0\}$.

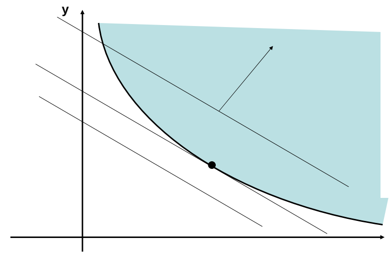
Since $\partial A \cap A \neq \emptyset$, the set A is not open. Furthermore, $\partial A \subset A$ so the set A is closed. Graphically, we see that A is not bounded. The set A is not compact (since is not bounded). We may show that the set A is convex in two different ways.

(i) Consider the function $g(x) = \frac{1}{x}$. It is easy to show that the function is convex. Therefore, the set $\{(x, y) \in \mathbb{R}^2 \in \mathbb{R} : x > 0, y \geq \frac{1}{x}\}$ is also convex.

(ii) Consider the function $g(x, y) = \ln(xy) = \ln x + \ln y$, defined on the convex set $D = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$. The Hessian matrix of this function is $\mathrm{H}\, g = \begin{pmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{pmatrix}$, which is negative definite. From here we conclude that the function g is concave in D. Since, $A = \{(x, y) \in D : g(x, y) \geq 0\}$, the set A is convex.

(b) We may no apply Weierstrass' Theorem since the set A is not compact. The level curves of $f(x, y) = x + 2y$ are sets of the form $\{(x, y) \in \mathbb{R}^2 : y = C - x/2\}$ which are straigt lines. Graphically (the vector indicates the direction of growth)





(c) Looking at the level curves of f we see that the funtion does not attain a (local o global) maximum on A. The global minimum is attained at the point of tangency of the straight line $y = C - x/2$ with the graph of $y = 1/x$,
This point satisfies that

$$-\frac{1}{2} = -\frac{1}{x^2}$$

that is $x = \pm\sqrt{2}$. And since $x > 0$, the minimum is attained at the point $\left(\sqrt{2}, 1/\sqrt{2}\right)$,