UNIVERSIDAD CARLOS III DE MADRID
MATHEMATICS II
EXERCISES

CHAPTER 1: Matrices and linear systems

1-1. *Compute the following determinants:*

$$a)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 0 & 5 \end{vmatrix} \qquad b)\begin{vmatrix} 3 & -2 & 1 \\ 3 & 1 & 5 \\ 3 & 4 & 5 \end{vmatrix} \qquad c)\begin{vmatrix} 1 & 2 & 4 \\ 1 & -2 & 4 \\ 1 & 2 & -4 \end{vmatrix}$$

1-2. *Use that* $\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = 25$, *to compute the value of* $\begin{vmatrix} 2a & 2c & 2b \\ 2u & 2w & 2v \\ 2p & 2r & 2q \end{vmatrix}$

1-3. *Verify the following identities without expanding the determinants:*

$$a)\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} \qquad b)\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$

1-4. *Solve the following equation, using the properties of the determinants.*

$$\begin{vmatrix} a & b & c \\ a & x & c \\ a & b & x \end{vmatrix} = 0$$

1-5. *Simplify and compute the following expressions.*

$$a)\begin{vmatrix} ab & 2b^2 & -bc \\ a^2c & 3abc & 0 \\ 2ac & 5bc & 2c^2 \end{vmatrix} \qquad b)\begin{vmatrix} x & x & x & x \\ x & a & a & a \\ x & a & b & b \\ x & a & b & c \end{vmatrix} \qquad c)\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

1-6. *Let A be a square matrix of order $n \times n$ such that $a_{ij} = i + j$. Compute $|A|$.*

1-7. *Let A be a square matrix of order $n \times n$ such that $A^t A = I$. Show that $|A| = \pm 1$.*

1-8. *Find the rank of the following matrices:*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{pmatrix}$$

1-9. *Study the rank of the following matrices, depending on the possible values of x.*

$$A = \begin{pmatrix} x & 0 & x^2 & 1 \\ 1 & x^2 & x^3 & x \\ 0 & 0 & 1 & 0 \\ 0 & 1 & x & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \qquad C = \begin{pmatrix} x & -1 & 0 & 1 \\ 0 & x & -1 & 1 \\ 1 & 0 & -1 & 2 \end{pmatrix}$$

1-10. *Let A and B be square, invertible matrices of the same order. Solve for X in the following equations.*
(a) $X^t \cdot A = B$.
(b) $(X \cdot A)^{-1} = A^{-1} \cdot B$.

Solve for X in the preceding equations, when $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

1-11. *Whenever possible, compute the inverse of the following matrices.*

$$A = \begin{pmatrix} 1 & 0 & x \\ -x & 1 & -\frac{x^2}{2} \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & x & 3 \\ 4 & 1 & -x \end{pmatrix}$$

1-12. *Whenever possible, compute the inverse of the following matrices,*

$$A = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

1-13. *Given the system* $\begin{cases} mx - y = 1 \\ x - my = 2m - 1 \end{cases}$ *compute* m *so that the system,*

(a) *has no solution,*
(b) *has infinitely many solutions,*
(c) *has a unique solution;* and
(d) *has a solution with* $x = 3$.

1-14. *Given the system of linear equations* $\begin{cases} x + ay = 1 \\ ax + z = 1 \\ ay + z = 2 \end{cases}$

(a) *Express it in matrix form;*
(b) *Write the unknowns, the independent terms and the associated homogeneous system ;*
(c) *Discuss and solve it according to the values of* a.

1-15. *Discuss and solve the following system* $\begin{cases} x + y + z + 2t - w = 1 \\ -x - 2y + 2w = -2 \\ x + 2z + 4t = 0 \end{cases}$

1-16. *Discuss and solve the following system according to the values of the parameters.*

$$\begin{cases} x + y + z = 0 \\ ax + y + z = b \\ 2x + 2y + (a+1)\,z = 0 \end{cases}$$

1-17. *Discuss and solve the following system according to the values of the parameters.*

$$\begin{cases} x - 2y + bz = 3 \\ 5x + 2y = 1 \\ ax + z = 2 \end{cases}$$

1-18. *Using Cramer's method, solve the following system.*

$$\begin{cases} -x + y + z = 3 \\ x - y + z = 7 \\ x + y - z = 1 \end{cases}$$

1-19. *Using Cramer's method, solve the following system.*

$$\begin{cases} x + y = 12 \\ y + z = 8 \\ x + z = 6 \end{cases}$$

1-20. *Using Cramer's method, solve the following system.*

$$\begin{cases} x + y - 2z = 9 \\ 2x - y + 4z = 4 \\ 2x - y + 6z = -1 \end{cases}$$

1-21. *Given the following system of two equations with three unknowns*

$$\begin{cases} x + 2y + z = 3 \\ ax + (a+3)y + 3z = 1 \end{cases}$$

(a) *Study for what values is of* a *the system is not compatible.*
(b) *For each value of the parameter* a*, for which the system is compatible, write the general solution.*

1-22. *Given the homogeneous system*

$$\begin{cases} 3x + 3y - z = 0 \\ -4x - 2y + mz = 0 \\ 3x + 4y + 6z = 0 \end{cases}$$

(a) *Compute* m *so that it has no trivial solutions and*
(b) *solve it for that value.*