University Carlos III
Department of Economics
Mathematics II. Final Exam. May 23rd 2024

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Given the following system of linear equations,

$$\begin{cases} x + 3y - az &= 4 \\ 2x - 3y + 2z &= 2 \\ 3x + az &= b \end{cases}$$

where $a, b \in \mathbb{R}$.

(a) **(20 points)** Classify the system according to the values of a and b.

(b) **(10 points)** Solve the above system for the values of a and b for which the system has infinitely many solutions.

(2) Consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : \quad y - x^2 + x \geq 0, \quad y - x - 3 \geq 0\}$$

and the function

$$f(x, y) = y - 2x$$

(a) **(20 points)** Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded, compact or convex.

(b) **(10 points)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A.

(c) **(10 points)** Draw the level curves of f, indicating the direction of growth of the function.

(d) **(20 points)** Using the level curves of f, determine (if they exist) the extreme global points of f on the set A.

(3) Consider the set of equations

$$\begin{aligned} 3xy + y^2 + z^2 &= 1 \\ x^2 + yz &= 1 \end{aligned}$$

(a) **(10 points)** Prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (1, 0, -1)$.

(b) **(20 points)** Compute

$$y'(x), \quad z'(x)$$

at the point $x_0 = 1$.

(c) **(20 points)** Compute

$$y''(x), \quad z''(x)$$

at the point $x_0 = 1$.

(4) Classify the following quadratic form $Q(x, y, z) = c^2x^2 - 2cxz + x^2 - 2xy - 2xz + y^2 + 2yz + 2z^2$ according to the values of $c \in \mathbb{R}$. **(30 points)**

(5) Consider the extreme points of the function

$$f(x, y) = x^2 - xy + y^2 - 3y$$

in the set

$$S = \left\{(x, y) \in \mathbb{R}^2 : 2x - y = 4\right\}$$

(a) **(10 points)** Write the Lagrangian function and the Lagrange equations.

(b) **(20 points)** Compute the solution(s) of the Lagrange equations.

(c) **(20 points)** Use the second order conditions to determine if the solution(s) of the Lagrange equations correspond to a local maximum or minimum value of f in S.

(d) **(20 points)** Does any of the solutions of the Lagrange equations correspond to global maximum or minimum of the function f in the set S?