University Carlos III
Department of Economics
Mathematics II. Final Exam. May 23rd 2024

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Given the following system of linear equations,

$$\begin{cases} x + 3y - az & = & 4 \\ 2x - 3y + 2z & = & 2 \\ 3x + az & = & b \end{cases}$$

where $a, b \in \mathbb{R}$.

(a) **(20 points)** Classify the system according to the values of a and b.

Solution: *The matrix associated with the system is*

$$\begin{pmatrix} 1 & 3 & -a & 4 \\ 2 & -3 & 2 & 2 \\ 3 & 0 & a & b \end{pmatrix}$$

We perform the following operations

$$\textit{row 2} \mapsto \textit{row 2} - 2 \times \textit{row 1}$$

$$\textit{row 3} \mapsto \textit{row 3} - 3 \times \textit{row 1}$$

And we obtain that the original system is equivalent to another one whose augmented matrix is the following

$$\begin{pmatrix} 1 & 3 & -a & 4 \\ 0 & -9 & 2a+2 & -6 \\ 0 & -9 & 4a & b-12 \end{pmatrix}$$

Now, we perform the operation row 3 $\mapsto$ *row 3* $-$ *row 2 and we obtain*

$$\begin{pmatrix} 1 & 3 & -a & 4 \\ 0 & -9 & 2a+2 & -6 \\ 0 & 0 & 2a-2 & b-6 \end{pmatrix}$$

We see that

(i) *if* $a \neq 1$, *then* $\operatorname{rank} A = 3 = \operatorname{rank}(A|b)$. *The system is consistent with a unique solution.*

(ii) *If* $a = 1$ *and* $b = 6$, *then* $\operatorname{rank} A = \operatorname{rank}(A|b) = 2$. *The system is consistent with* $3 - 2 = 1$ *parameters.*

(iii) *If* $a = 1$ *and* $b \neq 6$, *then* $\operatorname{rank} A = 2 < \operatorname{rank}(A|b) = 3$. *The system is not consistent.*

(b) **(10 points)** Solve the above system for the values of a and b for which the system has infinitely many solutions.

Solution: *We need* $a = 1$ *and* $b = 6$. *The proposed system of linear equations is equivalent to the following one*

$$\begin{cases} x + 3y - z & = & 4 \\ -9y + 4z & = & -6 \end{cases}$$

The solution is

$$z \in \mathbb{R}, \quad x = 2 - \frac{z}{3}, \quad y = \frac{2}{9}(2z + 3)$$

(2) Consider the set

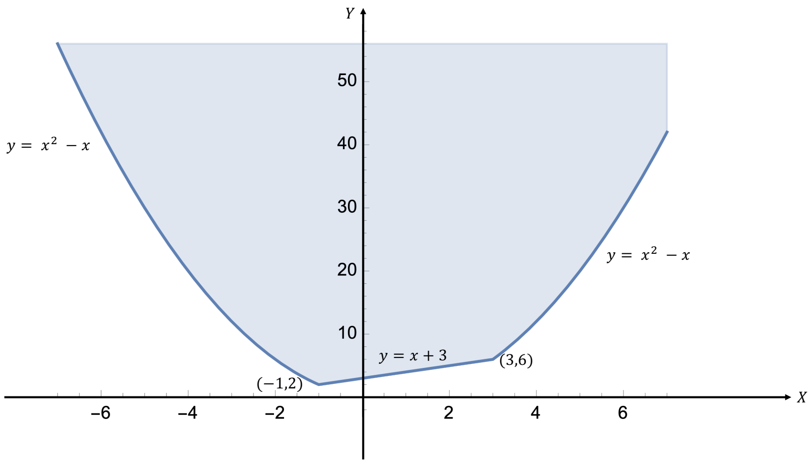
$$A = \{(x, y) \in \mathbb{R}^2 : \quad y - x^2 + x \geq 0, \quad y - x - 3 \geq 0\}$$

and the function

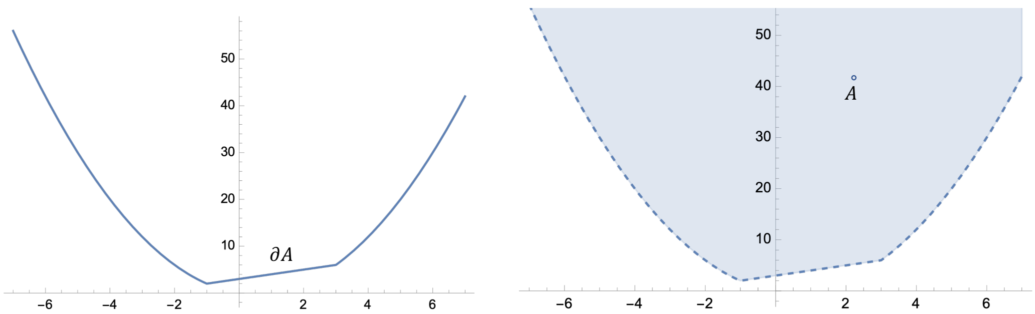
$$f(x, y) = y - 2x$$

(a) **(20 points)** Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded, compact or convex.

Solution: *The set A is approximately as indicated in the picture.*



The interior and the boundary are



The functions $h_1(x, y) = y - x^2 + x$ and $h_2(x, y) = y - x - 3$ are continuous (since, they are polynomials) and $A = \{(x, y) \in \mathbb{R}^2 : 1 \leq h_1(x, y) \geq 0, \; h_2(x, y) \geq 0\}$. Hence, the set A closed (Note also that $\partial A \subset A$). It is not open because $A \cap \partial A \neq \emptyset$.

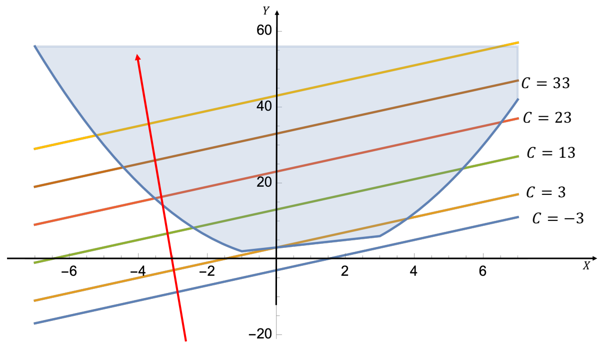
We see that any point of the form $(0, y)$ with $y \geq 10$ is in the set A. Hence, the set A is not bounded. Therefore, the set A is not compact.

We show next that A is also convex. The function $x^2 + x$ is convex. Hence the set $A_1 = \{(x, y) \in \mathbb{R}^2 : y \geq y - x^2 + x\}$ is convex. On he other hand, the function $x + 3$ is convex. Therefore, the set $A_2 = \{(x, y) \in \mathbb{R}^2 : y \geq x + 3\}$ is also convex. We conclude now that the set $A = A_1 \cap A_2$ is also convex.

(b) **(10 points)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A.

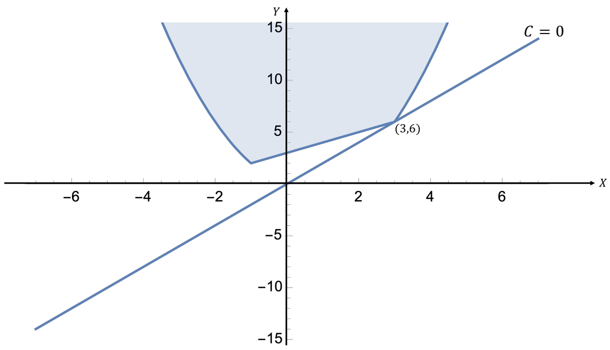
Solution: *The set A is not compact. Therefore, Weierstrass Theorem may not be applied.*

(c) **(10 points)** Draw the level curves of f, indicating the direction of growth of the function.
Solution:
The level curves $f(x,y) = y - 2x = C$ are straight lines of the form $y = 2x + C$ Graphically,



The red arrow represents the direction of growth of the function f.

(d) **(20 points)** Using the level curves of f, determine (if they exist) the extreme global points of f on the set A.
Solution: *Since, any point of the form $(0,y)$ with $y \geq 10$ is in the set A, the function f does not attain a maximum in A. Graphically, we wee that the minimum value is attained at the point $(3,6)$.*



The minimum value is $f(3,6) = 0$.

(3) Consider the set of equations

$$\begin{aligned} 3xy + y^2 + z^2 &= 1 \\ x^2 + yz &= 1 \end{aligned}$$

(a) **(10 points)** Prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (1, 0, -1)$.
Solution: *We first remark that* $(x_0, y_0, z_0) = (1, 0, -1)$. *is a solution of the system of equations. The functions* $f_1(x, y, z) = 3xy + y^2 + z^2 - 1$ *and* $f_2(x, y, z) = x^2 + yz - 1$ *are of class* C^∞. *We compute*

$$\left| \begin{matrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{matrix} \right|_{(x,y,z)=(1,0,-1)} = \left| \begin{matrix} 3x + 2y & 2z \\ z & y \end{matrix} \right|_{(x,y,z)=(1,0,-1)} = \left| \begin{matrix} 3 & -2 \\ -1 & 0 \end{matrix} \right| = -2$$

By the implicit function theorem, the above system of equations determines implicitly two differentiable functions $y(x)$ *and* $z(x)$ *in a neighborhood of the point* $(x_0, y_0, z_0) = (1, 0, -1)$.

(b) **(20 points)** Compute

$$y'(x), \quad z'(x)$$

at the point $x_0 = 1$.
Solution: *Differentiating implicitly with respect to* x,

$$2y(x)y'(x) + 3xy'(x) + 3y(x) + 2z(x)z'(x) = 0 \tag{1}$$
$$z(x)y'(x) + y(x)z'(x) + 2x = 0 \tag{2}$$
(3)

We plug in the values $x = 1$, $y(1) = 0$, $z(1)) = -1$ *to obtain the following*

$$\begin{aligned} 3y'(1) - 2z'(1) &= 0 \\ 2 - y'(1) &= 0 \end{aligned}$$

So,

$$y'(1) = 2, \quad z'(1) = 3$$

(c) **(20 points)** Compute

$$y''(x), \quad z''(x)$$

at the point $x_0 = 1$.
Solution: *Differentiation equation 1 with respect to* x *we obtain*

$$\begin{aligned} 3xy''(x) + 2y(x)y''(x) + 2y'(x)^2 + 6y'(x) + 2z(x)z''(x) + 2z'(x)^2 &= 0 \\ z(x)y''(x) + 2y'(x)z'(x) + y(x)z''(x) + 2 &= 0 \end{aligned}$$

We plug in the values $x = 1$, $y(1) = 0$, $z(1)) = -1$, $y'(1) = 2$, $z'(1) = 3$ *to obtain the following*

$$\begin{aligned} 3y''(1) - 2z''(1) + 38 &= 0 \\ 14 - y''(1) &= 0 \end{aligned}$$

So,

$$y''1) = 14, \quad z''(1) = 40$$

(4) Classify the following quadratic form $Q(x,y,z) = c^2x^2 - 2cxz + x^2 - 2xy - 2xz + y^2 + 2yz + 2z^2$ according to the values of $c \in \mathbb{R}$. **(30 points)**

Solution: *The associated matrix is*

$$A = \begin{pmatrix} c^2+1 & -1 & -c-1 \\ -1 & 1 & 1 \\ -c-1 & 1 & 2 \end{pmatrix}$$

We have $D_1 = c^2+1 > 0$. $D_2 = \begin{vmatrix} c^2+1 & -1 \\ -1 & 1 \end{vmatrix} = c^2 \geq 0$. *To compute* D_3 *we note that*

$$|A| = \begin{vmatrix} c^2+1 & -1 & -c-1 \\ -1 & 1 & 1 \\ -c-1 & 1 & 2 \end{vmatrix} \overset{r3 \mapsto r2-r3}{=} - \begin{vmatrix} c^2+1 & -1 & -c-1 \\ -1 & 1 & 1 \\ c & 0 & -1 \end{vmatrix} \overset{r2 \mapsto r2+r1}{=} - \begin{vmatrix} c^2+1 & -1 & -c-1 \\ c^2 & 0 & -c \\ c & 0 & -1 \end{vmatrix} = - \begin{vmatrix} c^2 & -c \\ c & -1 \end{vmatrix} = 0$$

So, $D_3 = |A| = 0$. *We see immediately that if* $c \neq 0$, *the quadratic form* Q *is positive semidefinite. If* $c = 0$, *The associated matrix is*

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

with $D_1 = 1 > 0$. $D_2 = D_3 = 0$. *However, if look at the chain of principal minors*

$$D_1 = a_{33} = 2 > 0, \quad D_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 > 0, \quad D_3 = 0$$

We see that the quadratic form is positive semidefinite.

(5) Consider the extreme points of the function

$$f(x,y) = x^2 - xy + y^2 - 3y$$

in the set

$$S = \{(x,y) \in \mathbb{R}^2 : 2x - y = 4\}$$

(a) **(10 points)** Write the Lagrangian function and the Lagrange equations.
Solution: *The Lagrangian is*

$$L(x,y) = x^2 - xy + y^2 - 3y + \lambda(2x - y - 4)$$

The Lagrange equations are

$$\begin{aligned} 2\lambda + 2x - y &= 0 \\ -\lambda - x + 2y - 3 &= 0 \\ 2x - y &= 4 \end{aligned}$$

(b) **(20 points)** Compute the solution(s) of the Lagrange equations.
Solution: *Plugging* $2x - y = 4$ *into the first equation we obtain* $\lambda = -2$. *Plugging now* $\lambda = -2$ *into the second equation we obtain the linear system*

$$\begin{aligned} -x + 2y &= 1 \\ 2x - y &= 4 \end{aligned}$$

whose solution is $x = 3$, $y = 2$.

(c) **(20 points)** Use the second order conditions to determine if the solution(s) of the Lagrange equations correspond to a local maximum or minimum value of f in S.
Solution: *The Hessian matrix associated with the Lagrangian is*

$$HL(x,y;\lambda) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

which is definite positive, since $D_1 = 2 > 0$ *and* $D_2 = 3 > 0$. *Hence the point* $(3,2)$ *corresponds to a local minimum.*

(d) **(20 points)** Does any of the solutions of the Lagrange equations correspond to global maximum or minimum of the function f in the set S?
Solution: *The set* S *is not compact. Therefore, Weiestrass' Theorem does not apply. However, we can check easily that the Hessian matrix of the function* $f(x,y) = x^2 - xy + y^2 - 3y$ *is also*

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

which, as seen above, is definite positive. Hence the function f *is convex in the (convex) set* S*.Therefore, the function* f *attains a minimum value on* S*, which must be a solution of the Lagrange equations. We conclude that the point* $(3,2)$ *corresponds to a global minimum of* f *on* S*. Since, we have seen in the previous part that the function* f *does not have a local maximum in* S*, we conclude immediately that it does not have neither a global maximum in* S*. Another way to obtain the same conclusion is to note that* $\lim_{y\to\infty} f(x, 2x-4) = \lim_{y\to\infty} \left(3x^2 - 18x + 28\right) = \infty$*, which also proves that the function* $f(x, 2x-4)$ *does not have a global maximum in* S*.*