**University Carlos III**
**Department of Economics**
**Mathematics II. Final Exam. May 19th 2023**

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|---|---|---|
| **Last Name:** | | **Name:** |
| **ID number:** | **Degree:** | **Group:** |

**IMPORTANT**

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

| Problem | Points |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| Total | |

(1) Given the following system of linear equations,

$$\begin{cases} -x + ay + 2z &= a \\ 2x + ay - z &= 2 \\ ax + 2z &= a \end{cases}$$

where $a \in \mathbb{R}$.

(a) **(20 points)** Classify the system according to the values of $a$.

(b) **(10 points)** Solve the above system for the values of $a$ for which it is consistent.

(2) Consider the set

$$A = \{(x,y) \in \mathbb{R}^2 : 2x + 10 \leq y \leq 10 - \frac{2x^2}{5}\}$$

and the function

$$f(x,y) = \sqrt{x^2 + y^2}$$

(a) **(20 points)** Sketch the graph of the set $A$, its boundary and its interior and justify if it is open, closed, bounded, compact or convex.

(b) **(10 points)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function $f$ defined on $A$.

(c) **(10 points)** Draw the level curves of $f$, indicating the direction of growth of the function.

(d) **(20 points)** Using the level curves of $f$, determine (if they exist) the extreme global points of $f$ on the set $A$.

(3) Consider the function $f(x,y) = 5x^3 - 2xy - x + 3y^2 - \frac{4y}{3}$.

(a) **(10 points)** Determine the critical points of the function $f$ in the set $\mathbb{R}^2$.
**Hint:** $\sqrt{784} = 28$.

(b) **(20 points)** Classify the critical points of the previous part into (local and/or global) maxima, minima and saddle points.

(c) **(10 points)** Find the largest open set of points in $\mathbb{R}^2$ where the function $f$ is convex.

(d) **(10 points)** Determine all the local/global solutions of the following problem

$$\begin{array}{ll} \max / \min & f(x,y) = 5x^3 - 2xy - x + 3y^2 - \frac{4y}{3} \\ \text{in the set} & A = \left\{(x,y) \in \mathbb{R}^2 : x > \frac{1}{4}\right\} \end{array}$$

(4) Consider the set of equations

$$\begin{aligned} x^3 + 5xy + z^2 &= 2 \\ xz + 2yz &= -1 \end{aligned}$$

(a) **(10 points)** Prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (1, 0, -1)$.

(b) **(20 points)** Compute

$$y'(x), \quad z'(x)$$

at the point $x_0 = 1$.

(c) **(10 points)** Compute Taylor's polynomial of order 1 of the functions $y(x)$ and $z(x)$ at the point $x_0 = 1$.

(5) Consider the extreme points of the function

$$f(x,y) = x^3 - x + y^2 + 2$$

in the set

$$S = \{(x,y,z) \in \mathbb{R}^3 : 2x^2 - 4x + 2y^2 = 0\}$$

(a) **(10 points)** Write the Lagrangian function and the Lagrange equations.

(b) **(20 points)** Compute the solution(s) of the Lagrange equations .

(c) **(20 points)** Use the second order conditions to determine if the solution(s) of the Lagrange equations correspond to a local maximum or minimum value of $f$ in $S$.

(d) **(10 points)** Does any of the solutions of the Lagrange equations correspond to global maximum or minimum of the function $f$ in the set $S$?