University Carlos III
Department of Economics
Mathematics II. Final Exam. May 19th 2023

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Given the following system of linear equations,

$$\begin{cases} -x + ay + 2z &= a \\ 2x + ay - z &= 2 \\ ax + 2z &= a \end{cases}$$

where $a \in \mathbb{R}$.

(a) **(20 puntos)** Classify the system according to the values of a.

Solution: *The matrix associated with the system is*

$$\begin{pmatrix} -1 & a & 2 & a \\ 2 & a & -1 & 2 \\ a & 0 & 2 & a \end{pmatrix}$$

We make the following elementary operations in the rows:

$$\textit{row 2} \mapsto \textit{row 2} + 2 \times \textit{row 1}$$

$$\textit{row 3} \mapsto \textit{row 3} + a \times \textit{row 1}$$

And we obtain that the original system is equivalent to another one whose augmented matrix is

$$\begin{pmatrix} -1 & a & 2 & a \\ 0 & 3a & 3 & 2+2a \\ 0 & a^2 & 2+2a & a+a^2 \end{pmatrix}$$

Now, we perform the operation row 3 $\mapsto$ *row 3* $- \frac{a}{3}$*row 2 and we obtain*

$$\begin{pmatrix} -1 & a & 2 & a \\ 0 & 3a & 3 & 2a+2 \\ 0 & 0 & a+2 & \frac{1}{3}a(a+1) \end{pmatrix}$$

We can see that,

(i) *if* $a \notin \{-2, 0\}$, *then* $\operatorname{rank} A = 3 = \operatorname{rank}(A|b) = 3$. *The system is consistent and determinate.*

(ii) *if* $a = 0$, *then the original system is equivalent to another one with augmented matrix*

$$\begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

with $\operatorname{rank} A = 2 < \operatorname{rank}(A|b) = 3$, *so it is an inconsistent system.*

(iii) *finally, if* $a = -2$, *then the original system is equivalent to another one with augmented matrix*

$$\begin{pmatrix} -1 & -2 & 2 & -2 \\ 0 & -6 & 3 & -2 \\ 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

with $\operatorname{rank} A = 2 < \operatorname{rank}(A|b) = 3$, *so it is also an inconsistent system.*

(b) **(10 puntos)** Solve the above system for the values of a for which it is consistent.

Solution: *The system is consistent and determinate if* $a \notin \{-2, 0\}$. *In this case the solutions are*

$$x = \frac{4+a}{3(a+2)}, \quad y = \frac{(a+1)(a+4)}{3a(a+2)}, \quad z = \frac{a(a+1)}{3(a+2)}$$

(2) Consider the set

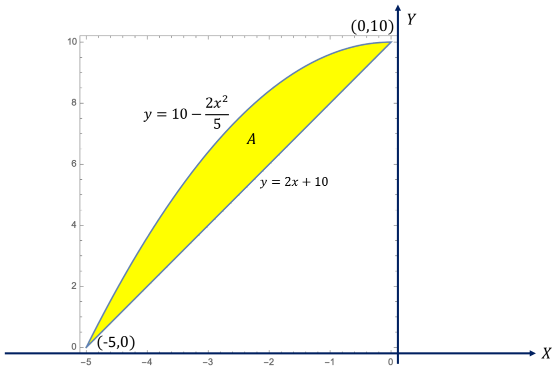
$$A = \{(x,y) \in \mathbb{R}^2 : 2x + 10 \le y \le 10 - \frac{2x^2}{5}\}$$

and the function

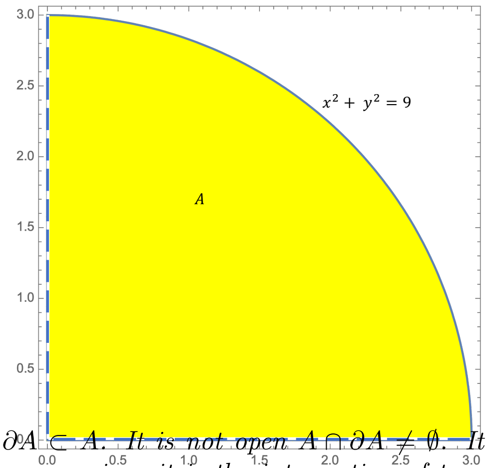
$$f(x,y) = \sqrt{x^2 + y^2}$$

(a) **(20 puntos)** Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded, compact or convex.

Solution: *The set A is approximately as indicated in the picture.*



The interior and the boundary are



The set A is closed since $\partial A \subset A$. It is not open $A \cap \partial A \neq \emptyset$. It is bounded aswell, so the set A is compact. The set is convex since it is the intersection of two convex sets, $A = B \cap C$ with $B = \{(x,y) \in \mathbb{R}^2 : 2x + 10 \le y\}$ y $C = \{(x,y) \in \mathbb{R}^2 : y \le 10 - \frac{2x^2}{5}\}$. The set B is a semiplane so is convex. The set C is a superlevel set of a concave function $g(x) = 10 - \frac{2x^2}{5}$ so is convex. Since A is the intersection of two convex sets it is also convex.

(b) **(10 puntos)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A.

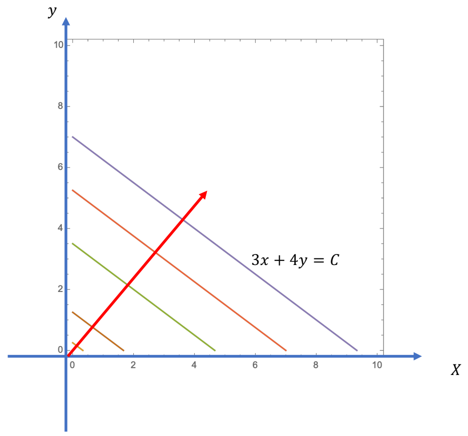
Solution: *The set A is compact. The function $f(x,y) = \sqrt{x^2 + y^2}$ is continuous in $\mathbb{R}^2$. Then f is continuous in every point of the set A and Weierstrass' theorem is fulfilled. The function f attains a global maximum and a global minimum on A.*

(c) **(10 puntos)** Draw the level curves of f, indicating the direction of growth of the function.

Solution: *For $D \ge 0$, the level curves $f(x,y) = \sqrt{x^2 + y^2} = D$ are circles*

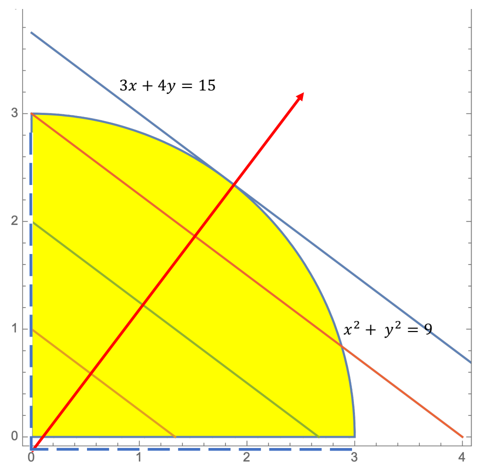
$$x^2 + y^2 = D^2 = C$$

Graphically,



In the picture we represent the level curves in several colours. The red arrow represents the direction of growth of the function f*.*

(d) **(20 puntos)** Using the level curves of f, determine (if they exist) the extreme global points of f on the set A.
Solution: *Graphically,*



we see that the maximum value is attained at the point $(0, 10)$*. The maximum value is* $f(0, 10) = 10$*. The minimum value is attained at the point* (a, b) *where the line* $y = 2x + 10$ *is tangent to the graph of the function* $y(x)$ *defined implicitly by* $x^2 + y^2 = C$*. At this point we have* $2x + 2yy' = 0$*. So,* $a + by'(a) = 0$*. On the other hand, the slope of the line* $y = 2x + 10$ *es* $m = 2$*. Hence,* $y'(a) = 2$ *and we obtain the linear system* $a + 2b = 0$, $b = 2a + 10$*. Therefore, the solution is* $a = -4$, $b = 2$*. The minimum value of* f *is attained at the point* $P = (-4, 2)$ *and* $f(-4, 2) = \sqrt{20} = 2\sqrt{5}$*.*

(3) Consider the function $f(x,y) = 5x^3 - 2xy - x + 3y^2 - \frac{4y}{3}$.

(a) **(10 puntos)** Determine the critical points of the function f in the set $\mathbb{R}^2$.
Hint: $\sqrt{784} = 28$.
Solution: *The gradient vector of the function f is*

$$\vec{\nabla} f(x,y) = \left(15x^2 - 2y - 1, -2x + 6y - \frac{4}{3}\right)$$

The equations that define the critical points are

$$\begin{aligned} 15x^2 - 2y - 1 &= 0 \\ -2x + 6y - \frac{4}{3} &= 0 \end{aligned}$$

The solutions are $\left(-\frac{13}{45}, \frac{17}{135}\right)$ *y* $\left(\frac{1}{3}, \frac{1}{3}\right)$.

(b) **(20 puntos)** Classify the critical points of the previous part into (local and/or global) maxima, minima and saddle points.
Solution: *The hessian matrix is*

$$\mathrm{H}(x,y) = \begin{pmatrix} 30x & -2 \\ -2 & 6 \end{pmatrix}$$

We calculate the value of the matrix at the critical points

$$\mathrm{H}\left(-\frac{13}{45}, \frac{17}{135}\right) = \begin{pmatrix} -\frac{26}{3} & -2 \\ -2 & 6 \end{pmatrix}, \quad \mathrm{H}\left(\frac{1}{3}, \frac{1}{3}\right) = \begin{pmatrix} 10 & -2 \\ -2 & 6 \end{pmatrix}$$

Therefore,

- *at the point* $\left(-\frac{13}{45}, \frac{17}{135}\right)$ *we obtain* $D_1 = -\frac{26}{3} < 0$, $D_2 = -56 < 0$. *The quadratic form is indefinite. Thus,* $\left(-\frac{13}{45}, \frac{17}{135}\right)$ *is a saddle point.*
- *at the point* $\left(\frac{1}{3}, \frac{1}{3}\right)$ *we obtain* $D_1 = 10 > 0$, $D_2 = 56 > 0$. *The quadratic form is positive definite. thus,* $\left(\frac{1}{3}, \frac{1}{3}\right)$ *corresponds to a strict local minimum.*

Moreover, we can see that $f(x,0) = 5x^3 - x$. *Then,* $\lim_{x\to\infty} f(0,x) = \infty$ *and* $\lim_{x\to-\infty} f(x,0) = -\infty$. *Hence there are no global extreme points.*

(c) **(10 puntos)** Find the largest open set of points in $\mathbb{R}^2$ where the function f is convex.
Solution: *We need to find the greatest open and convex set of points in* $\mathbb{R}^2$ *where the hessian matrix is positive definite or semidefinite for every point. This happens if* $D_1 = 30x > 0$ *and* $D_2 = 180x - 4 > 0$ *and the solution is* $S = \left\{(x,y) \in \mathbb{R}^2 : x > \frac{1}{45}\right\}$.

(d) **(10 puntos)** Determine all the local/global solutions of the following problem

$$\begin{array}{ll} \max/\min & f(x,y) = 5x^3 - 2xy - x + 3y^2 - \frac{4y}{3} \\ \text{in the set} & A = \left\{(x,y) \in \mathbb{R}^2 : x > \frac{1}{4}\right\} \end{array}$$

Solution: *We have seen in part (a) that the critical points are* $\left(-\frac{13}{45}, \frac{17}{135}\right)$ *and* $\left(\frac{1}{3}, \frac{1}{3}\right)$. *But only, the point* $\left(\frac{1}{3}, \frac{1}{3}\right)$ *satisfies the condition* $x > \frac{1}{4}$. *We have also seen that the function f is convex in the set of points* $S = \left\{(x,y) \in \mathbb{R}^2 : x > \frac{1}{45}\right\}$ *and since*

$$A = \left\{(x,y) \in \mathbb{R}^2 : x > \frac{1}{4}\right\} \subset S$$

we see that the function is convex in the set A. And we conclude that the critical point $\left(\frac{1}{3}, \frac{1}{3}\right)$ *corresponds to a global minimum of the problem.*

(4) Consider the set of equations

$$\begin{aligned} x^3 + 5xy + z^2 &= 2 \\ xz + 2yz &= -1 \end{aligned}$$

(a) **(10 puntos)** Prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (1, 0, -1)$.

Solution: *We first remark that $(x_0, y_0, z_0) = (1, 0, -1)$ is a solution of the system of equations. The functions $f_1(x, y, z) = x^3 + 5xy + z^2$ and $f_2(x, y, z) = xz + 2yz$ are of class C^∞. We compute*

$$\left| \begin{array}{cc} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{array} \right|_{(x,y,z)=(1,0,-1)} = \left| \begin{array}{cc} 5x & 2z \\ 2z & x+2y \end{array} \right|_{(x,y,z)=(1,0,-1)} = \left| \begin{array}{cc} 5 & -2 \\ -2 & 1 \end{array} \right| = 1 \neq 0.$$

By the implicit function theorem, the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (1, 0, -1)$.

(b) **(20 puntos)** Compute

$$y'(x), \quad z'(x)$$

at the point $x_0 = 1$.

Solution: *Differentiating implicitly with respect to x,*

$$\begin{aligned} 3x^2 + 5xy'(x) + 5y(x) + 2z(x)z'(x) &= 0 \\ 2z(x)y'(x) + 2y(x)z'(x) + xz'(x) + z(x) &= 0 \end{aligned}$$

We plug in the values $x = 1$, $y(1) = 0$, $z(1) = -1$ to obtain the following system

$$\begin{aligned} 5y'(1) - 2z'(1) + 3 &= 0 \\ -2y'(1) + z'(1) - 1 &= 0 \end{aligned}$$

So,

$$y'(1) = -1, \quad z'(1) = -1$$

(c) **(10 puntos)** Compute Taylor's polynomial of order 1 of the functions $y(x)$ and $z(x)$ at the point $x_0 = 1$.

Solution: *Taylor's polynomial of order 1 of the function $y(x)$ at the point x_0 is*

$$Q_1(x) = y(1) + y'(1)(x - x_0)$$

this is,

$$Q_1(x) = 0 - (x - 1) = 1 - x$$

Taylor's polynomial of order 1 of the function $z(x)$ at the point x_0 is

$$P_1(x) = z(1) + z'(1)(x - x_0)$$

so,

$$P_1(x) = -1 - (x - 1) = -x$$

(5) Consider the extreme points of the function

$$f(x, y) = x^3 - x + y^2 + 2$$

in the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : 2x^2 - 4x + 2y^2 = 0\}$$

(a) **(10 puntos)** Write the Lagrangian function and the Lagrange equations.
Solution: *The Lagrangian is*

$$L(x, y) = x^3 - x + y^2 + 2 + \lambda\left(2x^2 - 4x + 2y^2\right)$$

The Lagrange equations are

$$\begin{aligned} 3x^2 + \lambda(4x - 4) - 1 &= 0 \\ 4\lambda y + 2y &= 0 \\ 2x^2 - 4x + 2y^2 &= 0 \end{aligned}$$

(b) **(20 puntos)** Compute the solution(s) of the Lagrange equations .
Solution: *From the second equation we obtain* $(2\lambda + 1)y = 0$ *. Hence,* $y = 0$ *or* $\lambda = -\frac{1}{2}$*. Subsituting* $y = 0$ *in the third equation we obtain* $x^2 = 2x$*. Then,* $x = 0$ *or* $x = 2$*. We obtain the solutions*

$$x = 0, y = 0, \lambda = -\frac{1}{4}$$

and

$$x = 2, y = 0, \lambda = -\frac{11}{4}$$

Subsituting $\lambda = -\frac{1}{2}$ *in the first equation we obtain* $3x^2 - 2x + 1 = 0$*, but this equation has not real solutions.*

(c) **(20 puntos)** Use the second order conditions to determine if the solution(s) of the Lagrange equations correspond to a local maximum or minimum value of f in S.
Solution: *The Hessian matrix associated with the Lagrangian is*

$$HL(x, y; \lambda) = \begin{pmatrix} 4\lambda + 6x & 0 \\ 0 & 4\lambda + 2 \end{pmatrix}$$

Hence,

$$HL\left(0, 0; -\frac{1}{4}\right) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is indefinite. The associated quadratic form is $Q(x, y, z) = -x^2 + y^2$*. We compute the tangent space* $T_{(0,0)}S$*. Let* $g_1(x, y) = 2x^2 - 4x + 2y^2$*. Then,* $\nabla g_1(x, y) = (4x - 4, 4y)$*,* $\nabla g_1(0, 0) = (-4, 0)$*. Hence,*

$$T_{(0,0)}S = \{(x, y) \in \mathbb{R}^2 : (-4, 0) \cdot (x, y) = 0\} = \{(x, y) \in \mathbb{R}^2 : x = 0\} = \{(0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$$

Substituting $x = 0$ *in* $Q(x, y)$*, we obtain* $\bar{Q}(y) = Q(0, y) = y^2 > 0$ *if* $y \neq 0$*. Hence* $\bar{Q}(y)$ *is positive definite and the conditional critical point* $(0, 0)$ *corresponds to a local minimum of* f *on* S*. Furthermore,*

$$HL\left(2, 0; -\frac{11}{4}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -9 \end{pmatrix}$$

is indefinite. The associated quadratic form is $Q(x, y) = x^2 - 9y^2$*. Besides,* $\nabla g_1(2, 0) = (4, 0)$*, then,*

$$T_{(2,0)}S = \{(x, y) \in \mathbb{R}^2 : (4, 0) \cdot (x, y) = 0\} = \{(0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$$

Substituting $x = 0$*, in* $Q(x, y)$*, we obtain* $\bar{Q}(y) = Q(0, y) = -9y^2 < 0$ *if* $y \neq 0$*. Hence,* $\bar{Q}(z)$ *is negative definite and de conditional critical point* $(2, 0)$ *is a local maximum point of* f *in* S*.*

(d) **(10 puntos)** Does any of the solutions of the Lagrange equations correspond to global maximum or minimum of the function f in the set S?

Solution: *The constrain* $2x^2 - 4x + 2y^2 = 0$ *can be written like* $x^2 + (2-x)^2 + 2y^2 = 4$. *The set* S *is compact since* $0 \leq x \leq 2$, $-1 \leq y \leq 1$ *in* S. *Therefore, Weiestrass' Theorem can be applied and the function* f *attains a global maximum at* $(2,0)$ *and a global minimum at* $(0,0)$ *en* S. *The extreme vaules are* $f(2,0) = 8$ *y* $f(0,0) = 2$.

Another way of showing that the set S *is compact: the maximum value of the function* $4x - 2x^2$ *is* 2. *Then, in* S: $2y^2 = 4x - 2x^2 \leq 2$. *We can see that* $-1 \leq y \leq 1$. *But, in* S *we know that the function* $2y^2 = 4x - 2x^2$ *takes values in between* 0 *and* 2. *Hence* $0 \leq x \leq 2$. *The graph of the function* $4x - 2x^2$ *can help us to understand this.*