University Carlos III
Department of Economics
Mathematics II. Final Exam. May 20th 2022

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Given the following system of linear equations,

$$\begin{cases} 2x+3y+az &= 2a-1 \\ x+2y+z &= 1 \\ 5x+6y+(4a-3)z &= b \end{cases}$$

where $a, b \in \mathbb{R}$.

(a) **(10 points)** Classify the system according to the values of a and b.

(b) **(5 points)** Solve the above system for the values of a and b for which it is consistent.

(2) Consider the set

$$A = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 9,\ \ y \ge 0\}$$

and the function

$$f(x,y) = \frac{1}{5-x+y}$$

(a) **(10 points)** Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded, compact or convex.

(b) **(10 points)** Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A.

(c) **(5 points)** Draw the level curves of f, indicating the direction of growth of the function.

(d) **(5 points)** Using the level curves of f, determine (if they exist) the extreme global points of f on the set A.

(3) Consider the function $f(x,y) = 2x^3 - 6a^2x + 3y^2 - 2y^3 - 1$, with $a \in \mathbb{R}$, $a \neq 0$.

(a) **(10 points)** Determine the critical points of the function f in the set $\mathbb{R}^2$.

(b) **(10 points)** Classify the critical points of the previous part into (local and/or global) maxima and saddle points.

(c) **(5 points)** Find the greatest open set of points in $\mathbb{R}^2$ where the function f is convex.

(d) **(5 points)** Determine all the local/global solutions of the following problem

$$\begin{array}{rl} \max / \min & g(x,y) = 2x^3 - 24x + 3y^2 - 2y^3 - 1 \\ \text{in the set} & A = \{(x,y) \in \mathbb{R}^2 : x > 1, y < \frac{1}{4}\} \end{array}$$

(4) Consider the set of equations

$$\begin{aligned} t + xz^2 - 2y &= -5 \\ t^3 + x + y^2 - z &= 4 \end{aligned}$$

(a) **(5 points)** Prove that the above system of equations determines implicitly two differentiable functions $y(t,x)$ and $z(t,x)$ in a neighborhood of the point $(t_0, x_0, y_0, z_0) = (-1, 1, 2, 0)$.

(b) **(10 points)** Compute

$$\frac{\partial y}{\partial t}, \quad \frac{\partial y}{\partial x}, \quad \frac{\partial z}{\partial t}, \quad \frac{\partial z}{\partial x},$$

at the point $(-1, 1)$.

(c) **(5 points)** Compute Taylor's polynomial of order 1 of the function $z(t,x)$ at the point $(t_0, x_0) = (-1, 1)$.

(d) **(5 points)** Use Taylor's polynomial of order 1 of the function $z(t,x)$ at the point $(t_0, x_0) = (-1, 1)$ to estimate the value of $z(-0.9, 1.1)$.

(5) Consider the extreme points of the function

$$f(x,y) = xy - 3x - 6y$$

in the set

$$S = \{(x,y) : x + 2y = 20\}$$

(a) **(10 points)** Write the Lagrangian function and the Lagrange equations.

(b) **(5 points)** Compute the solution(s) of the Lagrange equations .

(c) **(10 points)** Use the second order conditions to determine if the solution(s) of the Lagrange equations correspond to a (local) maximum or minimum value of f on S.

(d) **(5 points)** Does any of the solutions of the Lagrange equations correspond to a global maximum or minimum value of f on S?