University Carlos III
Department of Economics
Mathematics II. Final Exam. May 20th 2022

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Given the following system of linear equations,

$$\begin{cases} 2x+3y+az &= 2a-1 \\ x+2y+z &= 1 \\ 5x+6y+(4a-3)z &= b \end{cases}$$

where $a, b \in \mathbb{R}$.

(a) **(10 points)** Classify the system according to the values of a and b.
Solution: *The matrix associated with the system is*

$$\begin{pmatrix} 2 & 3 & a & 2a-1 \\ 1 & 2 & 1 & 1 \\ 5 & 6 & 4a-3 & b \end{pmatrix}$$

Exchange rows 1 and 2. We obtain

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & a & 2a-1 \\ 5 & 6 & 4a-3 & b \end{pmatrix}$$

Next, we perform the following operations

$$\text{row } 2 \mapsto \text{row } 2 - 2 \times \text{row } 1$$

$$\text{row } 3 \mapsto \text{row } 3 - 5 \times \text{row } 1$$

And we obtain that the original system is equivalent to another one whose augmented matrix is the following

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & a-2 & 2a-3 \\ 0 & -4 & 4(a-2) & b-5 \end{pmatrix}$$

Now, we perform the operation row 3 $\mapsto$ row 3 $-$ 4row 2 and we obtain

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & a-2 & 2a-3 \\ 0 & 0 & 0 & -8a+b+7 \end{pmatrix}$$

We see that

(i) *if* $b \neq 8a-7$, *then* $\operatorname{rank} A = 2 < \operatorname{rank}(A|b) = 3$. *The system is not consistent.*

(ii) *If* $b = 8a-7$, *then* $\operatorname{rank} A = \operatorname{rank}(A|b) = 2$. *The system is consistent with* $3-2=1$ *parameters.*

(b) **(5 points)** Solve the above system for the values of a and b for which it is consistent.
Solution: *We need* $b = 8a-7$. *The proposed system of linear equations is equivalent to the following one*

$$\begin{cases} x+2y+z &= 1 \\ -y+(a-2)z &= 2a-3 \end{cases}$$

The solution is

$$z \in \mathbb{R}, \quad x = -5+4a+(3-2a)z, \quad y = 3-2a+(a-2)z$$

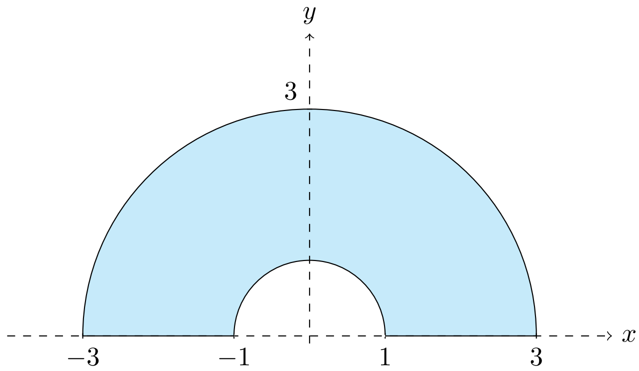
(2) Consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 9, \ \ y \geq 0\}$$

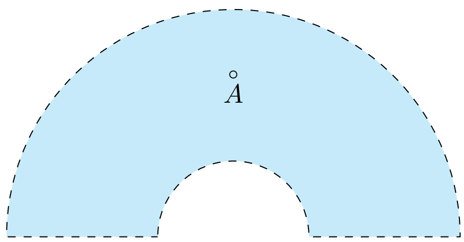
and the function

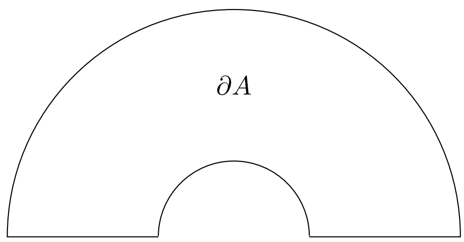
$$f(x, y) = \frac{1}{5 - x + y}$$

(a) **(10 points)** Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded, compact or convex.
Solution: *The set A is approximately as indicated (in blue) in the picture.*

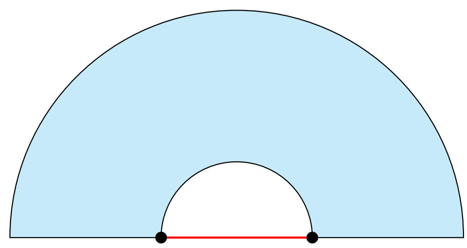


The interior and the boundary are





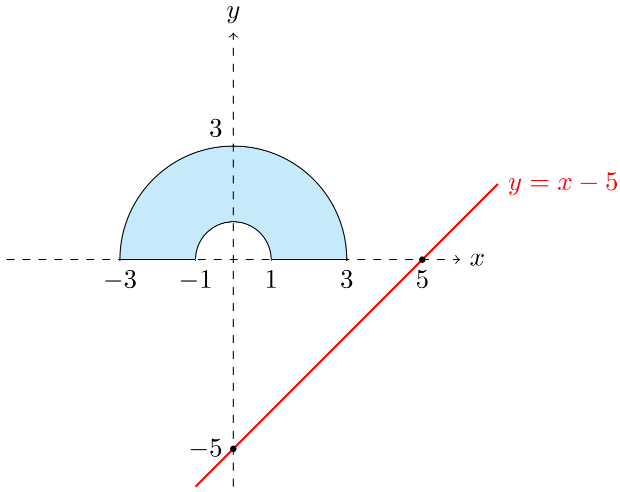
The functions $h_1(x, y) = x^2 + y^2$ and $h_2(x, y) = y$ are continuous and $A = \{(x, y) \in \mathbb{R}^2 : 1 \leq h_1(x, y) \leq 9, \ h_2(x, y) \geq 0\}$. Hence, the set A closed (Note also that $\partial A \subset A$). It is not open because $A \cap \partial A \neq \emptyset$.
We see that if $(x, y) \in A$, then $\|(x, y)\| \leq 9$. Hence the set A is bounded. Therefore, the set A is compact.
It is NOT convex because the points $(-1, 0), (1, 0) \in A$. But, the line segment that joins them is not contained in A.



(b) **(10 points)** Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A.
Solution: *The set A is compact. The function*

$$f(x, y) = \frac{1}{5 - x + y}$$

is continuous in its domain of definition $D(f) = \{(x, y) \in \mathbb{R}^2 : y - x \neq 5\}$. Note that the line $y = x - 5$ does not intersect the set A. Graphically,



Hence, the function f is continuous in all of A. Weierstrass Theorem applies and the function attains a global maximum and a global minimum on A.

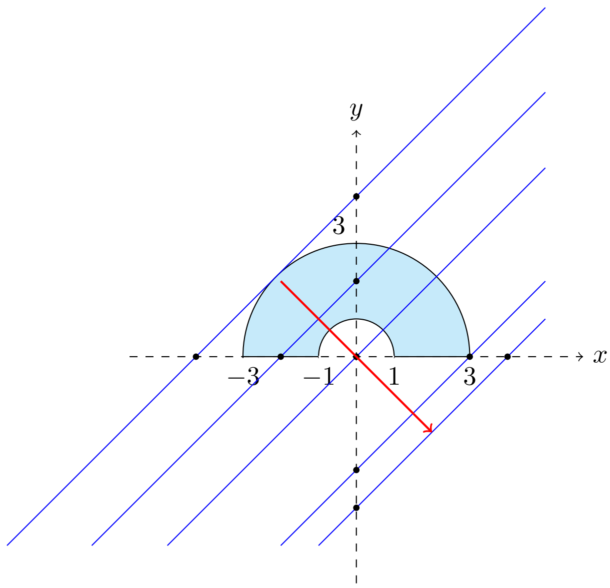
(c) **(5 points)** Draw the level curves of f, indicating the direction of growth of the function.
Solution: *The level curves*

$$f(x, y) = \frac{1}{5 - x + y} = D$$

are straight lines of the form

$$y = x + \frac{1}{D} - 5$$

Graphically,



In the picture we represent the level curves in blue color. The red arrow represents the direction of growth of the function f.

(d) **(5 points)** Using the level curves of f, determine (if they exist) the extreme global points of f on the set A.
Solution: *Graphically, we see that the maximum value is attained at the point $(3, 0)$. The maximum value is $f(3, 0) = 1/2$.*
The minimum value is attained at the point (a, b) where the line $y = x + \frac{1}{D} - 5$ is tangent to the graph of the function $y(x)$ defined implicitly by $x^2 + y^2 = 9$. At this point we have $2x + 2yy' = 0$. So, $y'(x) = -x/y(t, x)$. At the point $x = a$, $y(a) = b$, we have $y'(a) = -a/b$. On the other, hand the slope of the line $y = x + \frac{1}{D} - 5$ is $m = 1$. hence $a = -b$ and we must have $2a^2 = 9$ or $a = -\frac{3}{\sqrt{2}}$.
The minimum value is

$$f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \frac{1}{5 + 2\frac{3}{\sqrt{2}}}$$

(3) Consider the function $f(x,y) = 2x^3 - 6a^2x + 3y^2 - 2y^3 - 1$, with $a \in \mathbb{R}$, $a \neq 0$.

(a) **(10 points)** Determine the critical points of the function f in the set $\mathbb{R}^2$.

Solution: *The gradient vector of the function f is*

$$\vec{\nabla} f(x,y) = \left(6x^2 - 6a^2, 6y - 6y^2\right)$$

The equations that define the critical points are

$$\begin{aligned} 6x^2 &= 6a^2 \\ 6y &= 6y^2 \end{aligned}$$

The solutions are $(a,0)$, $(-a,0)$, $(a,1)$ and $(-a,1)$.

(b) **(10 points)** Classify the critical points of the previous part into (local and/or global) maxima and saddle points.

Solution: *The hessian matrix is*

$$\mathrm{H}(x,y) = \begin{pmatrix} 12x & 0 \\ 0 & 6-12y \end{pmatrix}$$

We calculate the value of the matrix at the critical points

$$\mathrm{H}(a,0) = \begin{pmatrix} 12a & 0 \\ 0 & 6 \end{pmatrix}, \quad \mathrm{H}(-a,0) = \begin{pmatrix} -12a & 0 \\ 0 & 6 \end{pmatrix},$$

$$\mathrm{H}(a,1) = \begin{pmatrix} 12a & 0 \\ 0 & -6 \end{pmatrix}, \quad \mathrm{H}(-a,1) = \begin{pmatrix} -12a & 0 \\ 0 & -6 \end{pmatrix}$$

Therefore,

- *The point $(a,0)$ corresponds to a local minimimum if $a > 0$ and saddle point if $a < 0$;*
- *The point$(-a,0)$ is a saddle point if $a > 0$ and corresponds to a local minimum if $a < 0$;*
- *The point $(a,1)$ is a saddle point if $a > 0$ and corresponds to a local maximum if $a < 0$;*
- *The point $(-a,1)$ corresponds to a local maximum if $a > 0$ and saddle point if $a < 0$.*

On the other hand, we can see that $f(0,y) = 3y^2 - 2y^3 - 1$. Then $\lim_{y\to\infty} f(0,y) = -\infty$ and $\lim_{y\to-\infty} f(0,y) = +\infty$. Hence there are no global extreme points.

(c) **(5 points)** Find the greatest open set of points in $\mathbb{R}^2$ where the function f is convex.

Solution: *We need to find the greatest open and convex set of points in $\mathbb{R}^2$ where the hessian matrix is positive definite or semidefinite for every point. This happens if $D_1 = 12x \geq 0$ and $D_2 = 12x(6-12y) \geq 0$ and the solution is*

$$S = int\left(\{(x,y) \in \mathbb{R}^2 : x \geq 0; \frac{1}{2} \geq y\}\right) = \{(x,y) \in \mathbb{R}^2 : x > 0, y < \frac{1}{2}\}$$

(d) **(5 points)** Determine all the local/global solutions of the following problem

$$\begin{aligned} &\max / \min \quad g(x,y) = 2x^3 - 24x + 3y^2 - 2y^3 - 1 \\ &\text{in the set} \quad A = \{(x,y) \in \mathbb{R}^2 : x > 1, y < \tfrac{1}{4}\} \end{aligned}$$

Solution: *Note that the set A is convex. The function g is obtained from the function f by taking $a = 2$. From part (a) the critical points are $(2,0)$, $(-2,0)$, $(2,1)$ and $(-2,1)$. Only the point $(2,0)$ satisfies the constraints $x > 1$, $y < \frac{1}{4}$. On the other hand, by part (c) the function g is convex in the set $S = \{(x,y) \in \mathbb{R}^2 : x > 0; y < \frac{1}{2}\}$. Since,*

$$A = \{(x,y) \in \mathbb{R}^2 : x > 1, y < \frac{1}{4}\} \subset S = \{(x,y) \in \mathbb{R}^2 : x > 0, y < \frac{1}{2}\}$$

we see that the function g is convex in the set A. And we conclude that the critical point $(2,0)$ corresponds to a global minimum of the problem.

(4) Consider the set of equations

$$\begin{aligned} t + xz^2 - 2y &= -5 \\ t^3 + x + y^2 - z &= 4 \end{aligned}$$

(a) **(5 points)** Prove that the above system of equations determines implicitly two differentiable functions $y(t, x)$ and $z(t, x)$ in a neighborhood of the point $(t_0, x_0, y_0, z_0) = (-1, 1, 2, 0)$.
Solution: *We first remark that* $(t_0, x_0, y_0, z_0) = (-1, 1, 2, 0)$. *is a solution of the system of equations. The functions* $f_1(t, x, y, z) = t + xz^2 - 2y - 5$ *and* $f_2(x, y, z) = t^3 + x + y^2 - z - 4$ *are of class* C^∞. *We compute*

$$\left| \begin{matrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{matrix} \right|_{(t,x,y,z)=(-1,1,2,0)} = \left| \begin{matrix} -2 & 2xz \\ 2y & -1 \end{matrix} \right|_{(t,x,y,z)=(-1,1,2,0)} = \left| \begin{matrix} -2 & 0 \\ 4 & -1 \end{matrix} \right| = 2$$

By the implicit function theorem, the above system of equations determines implicitly two differentiable functions $y(t, x)$ *and* $z(t, x)$ *in a neighborhood of the point* $(t_0, x_0, y_0, z_0) = (-1, 1, 2, 0)$.

(b) **(10 points)** Compute

$$\frac{\partial y}{\partial t}, \quad \frac{\partial y}{\partial x}, \quad \frac{\partial z}{\partial t}, \quad \frac{\partial z}{\partial x},$$

at the point $(-1, 1)$.
Solution: *Differentiating implicitly with respect to* x,

$$\begin{aligned} -2\frac{\partial y}{\partial x}(t, x) + 2x\frac{\partial z}{\partial x}(t, x)z(t, x) + z(t, x)^2 &= 0 \\ 2y(t, x)\frac{\partial y}{\partial x}(t, x) - \frac{\partial z}{\partial x}(t, x) + 1 &= 0 \end{aligned}$$

We plug in the values $(t, x) = (-1, 1)$, $(y(-1, 1), z(-1, 1)) = (2, 0)$ *to obtain the following*

$$\begin{aligned} -2\frac{\partial y}{\partial x}(-1, 1) &= 0 \\ 4\frac{\partial y}{\partial x}(-1, 1) - \frac{\partial z}{\partial x}(-1, 1) + 1 &= 0 \end{aligned}$$

So,

$$\frac{\partial y}{\partial x}(-1, 1) = 0, \quad \frac{\partial z}{\partial x}(-1, 1) = 1$$

Differentiating now implicitly with respect to t,

$$\begin{aligned} -2\frac{\partial y}{\partial t}(t, x) + 2xz(t, x)\frac{\partial z}{\partial t}(t, x) + 1 &= 0 \\ 2y(t, x)\frac{\partial y}{\partial t}(t, x) - \frac{\partial z}{\partial t}(t, x) + 3t^2 &= 0 \end{aligned}$$

We plug in the values $(t, x) = (-1, 1)$, $(y(-1, 1), z(-1, 1)) = (2, 0)$ *to obtain the following*

$$\begin{aligned} 1 - 2\frac{\partial y}{\partial t}(-1, 1) &= 0 \\ 4\frac{\partial y}{\partial t}(-1, 1) - \frac{\partial z}{\partial t}(-1, 1) + 3 &= 0 \end{aligned}$$

So,

$$\frac{\partial y}{\partial t}(-1, 1) = \frac{1}{2}, \quad \frac{\partial z}{\partial t}(-1, 1) = 5$$

(c) **(5 points)** Compute Taylor's polynomial of order 1 of the function $z(t, x)$ at the point $(t_0, x_0) = (-1, 1)$.
Solution: *Taylor's polynomial of order 1 of the function* $z(t, x)$ *at the point* (t_0, x_0) *is*

$$P_1(t, x) = z(-1, 1) + \frac{\partial z}{\partial t}(-1, 1)(t - t_0) + \frac{\partial z}{\partial x}(-1, 1)(x - x_0)$$

That is,

$$P_1(t,x) = 0 + 5(t+1) + (x-1) = 5(t+1) + x - 1$$

(d) **(5 points)** Use Taylor's polynomial of order 1 of the function $z(t,x)$ at the point $(t_0,x_0) = (-1,1)$ to estimate the value of $z(-0.9, 1.1)$.
Solution: $z(-0.9,1.1) \approx P_1(-0.9,1.1) = 5 \times 0.1 + 0.1 = 0.6.$

(5) Consider the extreme points of the function

$$f(x,y) = xy - 3x - 6y$$

in the set

$$S = \{(x,y) : x + 2y = 20\}$$

(a) **(10 points)** Write the Lagrangian function and the Lagrange equations.
Solution: *The Lagrangian is*

$$L(x,y) = xy - 3x - 6y + \lambda(20 - x - 2y)$$

The Lagrange equations are

$$\begin{aligned} y - 3 - \lambda &= 0 \\ x - 6 - 2\lambda &= 0 \\ 20 - 2x - 2y &= 0 \end{aligned}$$

(b) **(5 points)** Compute the solution(s) of the Lagrange equations .
Solution: *Multiplying the first equation by* 2 *and comparing with the second equation we obtain* $x = 2y$. *Subsituting* $x = 2y$ *into the third equation, we obtain* $2x = 4y = 20$. *Hence the solution is*

$$x = 10, \quad y = 5, \lambda = 2$$

(c) **(10 points)** Use the second order conditions to determine if the solution(s) of the Lagrange equations correspond to a (local) maximum or minimum value of f on S.
Solution: *The Hessian matrix associated with the Lagrangian is*

$$HL(x,y;\lambda) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which is indefinite. The associated quadratic form is

$$Q(x,y) = 2xy$$

We compute the space $T_{(10,5)}S$. *Let* $g(x,y) = x + 2y - 20$. *We have,* $\nabla g(x,y) = (1,2)$, $\nabla g(10,5) = (1,2)$. *Hence,*

$$T_{(10,5)}S = \{(x,y) \in \mathbb{R}^2 : x + 2y = 0\} = \{(-2y, y) : y \in \mathbb{R}\}$$

Substituting $x = -2y$ *in* $Q(x,y)$, *we obtain* $\bar{Q}(y) = Q(-2y,y) = -4y^2 < 0$ *if* $y \neq 0$. *Hence* $\bar{Q}(y)$ *is negative definite and the point* $(10,5)$ *corresponds to a local maximum of* f *on* S.

(d) **(5 points)** Does any of the solutions of the Lagrange equations correspond to a global maximum or minimum value of f on S?
Solution: *The set* S *is not compact. Therefore, Weiestrass' Theorem does not apply. However, substituting the restriction* $x = 20-2y$ *into the function, we obtain* $f(20-2y,y) = -2y^2+20y-60$. *This is a concave function with a unique global maximum. Furthermore, since* $\lim_{y\to\infty} f(20-2y,y) = -\infty$, *the function* $f(20-2y,y)$ *does not have a global minimum. We conclude that the point* $(10,5)$ *corresponds to a global maximum of* f *on* S.