Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	Total
Points						

Department of Economics **Mathematics I Extraordinary Final Exam** **June 23rd 2022**

Exam time: 2 hours.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Let** $C(x) = b + 16x + 4x^2$ **be the cost function and** $p(x) = a - x$ **the inverse demand function of a monopolistic firm, with** $a, b > 0$**. Then:**

(a) calculate the value of the parameter a, knowing that the production level to maximize the profit is $x^* = 5$.

(b) calculate the value of the parameter b, knowing that the production level to maximize the profit per unit is $x^{**} = 4$.

0.5 points part a); 0.5 points part b).

(a) First of all, we calculate the profit function.
$B(x) = (a - x)x - (b + 16x + 4x^2) = -5x^2 + (a - 16)x - b$
Secondly, we calculate the first and second order derivatives of B:
$B'(x) = -10x + a - 16; B''(x) = -10 < 0$
we see that B has an unique critical point at $x^* = \dfrac{a-16}{10}$ and, since B is a concave function, the critical point is the global maximizer.
Finally $x^* = 5 = \dfrac{a-16}{10} \Longrightarrow a - 16 = 50 \Longrightarrow a = 66.$

(b) The profit per unit function is $\dfrac{B(x)}{x} = -5x + (a - 16) - \dfrac{b}{x}$,
we calculate its first and second order derivative functions: $\left(\dfrac{B(x)}{x}\right)' = -5 + \dfrac{b}{x^2} = 0 \Longleftrightarrow x^2 = \dfrac{b}{5}$.
Since $\left(\dfrac{B(x)}{x}\right)'' = -\dfrac{2b}{x^3} < 0$, The function is concave and the critical point is the global maximizer.
Then $x^{**} = 4 = \sqrt{\dfrac{b}{5}} \Longrightarrow b = 80.$

(2) **Given the implicit function $y = f(x)$, defined by the equation $e^{x+y} + xy^2 = e$ in a neighbourhood of the point $x = 1, y = 0$, it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function f at $a = 1$.

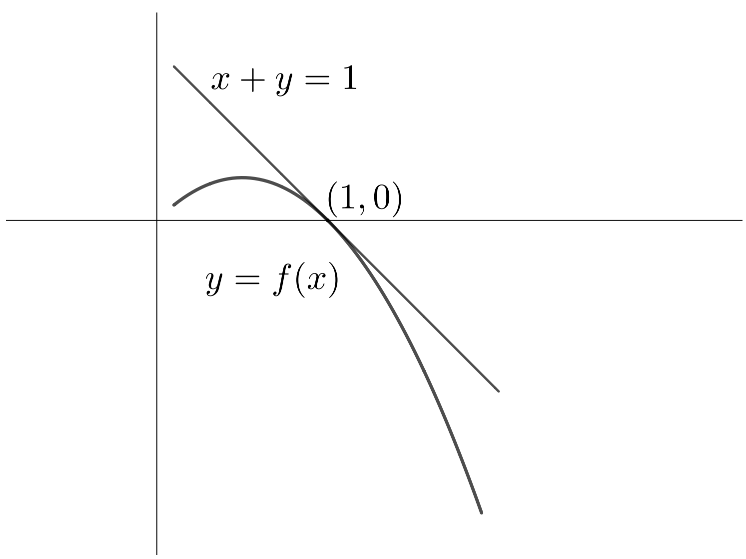
(b) sketch the graph of the function f near the point $x = 1, y = 0$.

(c) use second-order Taylor Polynomial of $f(x)$ to obtain the approximate values of $f(0,9)$ and $f(1,2)$. Use this polynomial to compare $f(1)$ with $\frac{2}{3}f(0,9) + \frac{1}{3}f(1,2)$.
(*Hint for parts (b) and (c):* use that $f''(1) < 0$).
0.4 points part a); 0.2 points part b) 0.4 points part c).

(a) First of all, we calculate the first-order derivative of the equation:
$e^{x+y}(1 + y') + y^2 + 2xyy' = 0$
evaluating at $x = 1,\ y(1) = 0$ we obtain: $y'(1) = f'(1) = -1$.
Then the equation of the tangent line is: $y = P_1(x) = -(x - 1)$ o $x + y = 1$.
Secondly, we calculate the second-order derivative of the equation:
$e^{x+y}[(1 + y')^2 + y''] + 2yy' + 2yy' + 2x(y')^2 + 2xyy'' = 0$
evaluating at $x = 1, y(1) = 0, y'(1) = -1$ we obtain $y''(1) = f''(1) = -2/e$.
Therefore, the second-order Taylor Polynomial is: $y = P_2(x) = -(x - 1) - \frac{1}{e}(x - 1)^2$.

(b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 1$, will be as you can see in the figure underneath.

(c) On the other hand, using this Taylor Polynomial, we obtain:
$f(0,9) \approx 0,1 - \dfrac{1}{e}\, 0,01;\ f(1,2) \approx -0,2 - \dfrac{1}{e} 0,04 \Longrightarrow$
$\frac{2}{3}f(0,9) + \frac{1}{3}f(1,2) = -\dfrac{1}{e}0,02 < 0 = f(1) = f(\frac{2}{3}0,9 + \frac{1}{3}1,2)$.
And this is reasonable since, $f(x)$ is concave function near $x = 1$.



(3) **Consider the function $f(x) = \dfrac{\sqrt{x^2+1}}{x+1}$. Then:**

(a) find the domain and the asymptotes of function $f(x)$.

(b) find the intervals where $f(x)$ increases and decreases and its range. Draw the graph of the function.

(c) consider $f_1(x)$ to be the function $f(x)$ defined on the interval $[0, \infty)$. Find, if they exist, the global extreme points of $f_1(x)$.

0.4 points part a); 0.4 points part b); 0.2 points part c)

(a) First of all, the domain of the function is $\mathbb{R} - \{-1\}$.

If we calculate the right-hand sided limit at $x = -1$, we obtain $\lim_{x\to -1^+} \dfrac{\sqrt{x^2+1}}{x+1} = \dfrac{\sqrt{2}}{0^+} = \infty$.

Analogously, we calculate the left-hand sided limit at the point, $\lim_{x\to -1^-} \dfrac{\sqrt{x^2+1}}{x+1} = \dfrac{\sqrt{2}}{0^-} = -\infty$.

Therefore, $f(x)$ has a vertical asymptote at $x = -1$.

Secondly, to find horizontal asymptotes we calculate the limit towards ∞, to obtain $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x+1}$ =(dividing the numerator and denominator by x) = = $\lim_{x\to\infty} \dfrac{\sqrt{1+1/x^2}}{1+1/x} = 1$.

Then f has an horizontal asymptote $y = 1$ at ∞.

Moreover, we calculate the limit at $-\infty$ of f and we obtain $\lim_{x\to-\infty} \dfrac{\sqrt{x^2+1}}{x+1}$ =(dividing the numerator and denominator by $-x$, that we introduce inside the square root as $1/x^2$) = $\lim_{x\to\infty} \dfrac{\sqrt{1+1/x^2}}{-1-1/x} =$ -1.

Then, f has an horizontal asymptote $y = -1$ at $-\infty$. Obviously, because there are both horizontal asymptotes then oblique asymptotes do not exist.

(b) In order to study the monotonicity of the function, we calculate the sign of its derived function:

$f'(x) = (\dfrac{\sqrt{x^2+1}}{x+1})' = \dfrac{(2x/2\sqrt{x^2+1})(x+1) - \sqrt{x^2+1}}{(x+1)^2} = \dfrac{x(x+1)-(x^2+1)}{(x+1)^2\sqrt{x^2+1}} =$

$= \dfrac{x-1}{(x+1)^2\sqrt{x^2+1}}$, since the denominator is always positive the sign of the derived function is calculated by the numerator $x - 1$, and we concluded that:

i) $f'(x) > 0 \Leftrightarrow x \in (1, \infty)$, then f is increasing on $[1, \infty)$.

ii) $f'(x) < 0 \Leftrightarrow x \in (-\infty, -1) \cup (-1, 1)$, then f is decreasing on $(-\infty, -1)$ and $(-1, 1)$.

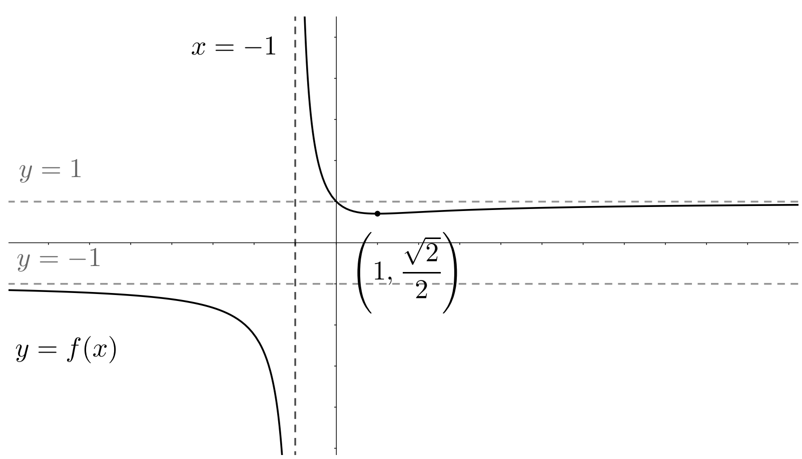
To find the range, since $f(x)$ is continuous in its domain and using the intermediate value theorem we say:

i) the range of the interval $(-\infty, -1)$ is $(-\infty, -1)$.

ii) the range of the interval $(-1, \infty)$, taking into account that $f(1) = \dfrac{\sqrt{2}}{2}$, is $[\dfrac{\sqrt{2}}{2}, \infty)$.

Thus, the range of f is $(-\infty, -1) \cup [\dfrac{\sqrt{2}}{2}, \infty)$.

The graph of $f(x)$ will have an appearance approximately, similar to this one:



(c) About the global extreme points of f_1, $x = 1$ is the global minimizer, since f_1 is decreasing on $[0, 1]$ and increasing on $[1, \infty)$.

On the other hand, $x = 0$ is the global maximizer of $f_1(x)$ since, f_1 is decreasing on $[0, 1]$ and increasing on $[1, \infty)$ and $f_1(x)$ has the horizontal asymptote $y = 1 = f_1(0)$, we can confirm that $f_1(x) \leq 1 = f_1(0)$.

(4) **Let**

$$f(x) = \begin{cases} \dfrac{\ln(x^2+1)}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

you are asked:

(a) prove that the function is derivable at $x = 0$.

(b) find the asymptotes of the function.

(c) consider $f_1(x)$ to be the function $f(x)$ defined on the interval $[0, \infty)$. Find the global minimum of this function. Study if this function attains its global maximum. (*Hint:* You only need to prove if the global maximum exists or not.)

0.4 points part a); 0.2 points part b); 0.4 points part c)

(a) To begin with, we study if the function is continuous $x = 0$.

$\lim\limits_{x \longrightarrow 0} f(x) = \lim\limits_{x \longrightarrow 0} \dfrac{\ln(x^2+1)}{x} = \frac{0}{0}$ =(L'Hopital)= $\lim\limits_{x \longrightarrow 0} \dfrac{2x}{x^2+1} = 0$, then it is continuous at $x = 0$.

Now, we study if the function is derivable at the same point, since it is continuous, we need to prove the existence of the limit:

$\lim\limits_{x \longrightarrow 0} f'(x) = \lim\limits_{x \longrightarrow 0} \dfrac{[2x/(x^2+1)]x - \ln(x^2+1)}{x^2} = \lim\limits_{x \longrightarrow 0} \dfrac{2x^2/(x^2+1)}{x^2} - \lim\limits_{x \longrightarrow 0} \dfrac{\ln(x^2+1)}{x^2}$.

Obviously the first limit is equal to 2. And we calculate the second:

$\lim\limits_{x \longrightarrow 0} \dfrac{\ln(x^2+1)}{x^2} = \frac{0}{0}$ =(L'Hopital)= $\lim\limits_{x \longrightarrow 0} \dfrac{2x/(x^2+1)}{2x} = 1$.

Then, we can say that $f'(0) = 2 - 1 = 1$.

(b) Since the function is continuous in its domain there are not any vertical asymptotes.

About asymptotes at infinitum:

$\lim\limits_{x \longrightarrow \infty} \dfrac{\ln(x^2+1)}{x} = \frac{\infty}{\infty}$ =(L'Hopital)= $\lim\limits_{x \longrightarrow \infty} \dfrac{2x}{x^2+1} = 0$, then there is a horizontal asymptote: $y = 0$.

Analogously, $y = 0$ is the asymptote at $-\infty$.

(c) Since $f(x) > 0$ if $x > 0$ (because $\ln(1+x^2) > \ln 1 = 0$, when $x > 0$),

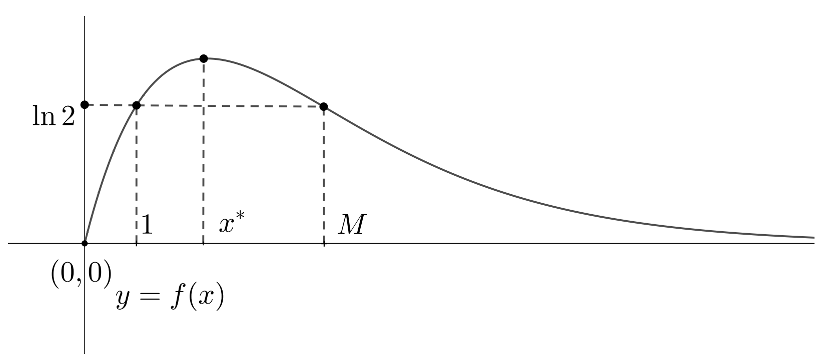
we can say that $x = 0$ is the global minimizer and $f(0) = 0$ is the global minimum.

The global maximum also exists, as the function is continuous, $\lim\limits_{x \longrightarrow \infty} f(x) = 0$ and given $f(1) = \ln 2 > 0$, we can find $M > 0$ such that, $f(x) < f(1)$ if $x > M$.

Now, using Weierstrass' Theorem to f in the interval $[0, M]$, we know that exists x^* maximizer of f in the interval.

Obviously, x^* is also the maximizer of f in $[0, \infty)$.

You can observe this in the following graph:



(5) **Given the functions** $f, g : [-1,1] \longrightarrow \mathbb{R}$, **defined by:** $f(x) = e^{-2x}$, $g(x) = e^{-x/2}$, **then:**

(a) draw approximately the set A, bounded by the graph of these functions and the straight lines $x = -1$,$x = 1$. Find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) calculate the area of the given set.

Hint for part (a): Pareto order is defined as: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1$.

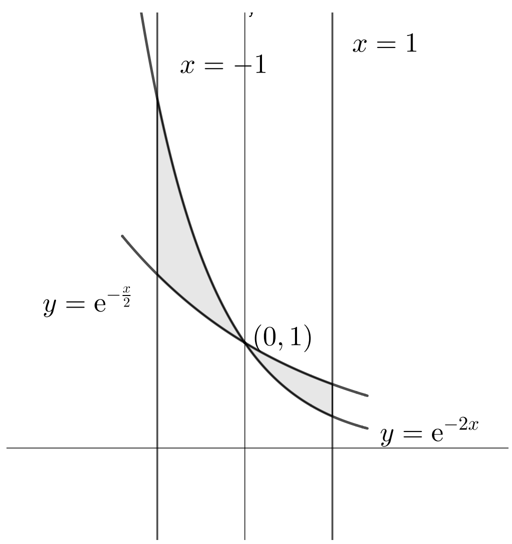
0.6 points part a); 0.4 points part b).

(a) First of all, we can notice $f(0) = g(0) = 1$. Secondly:

i) if $x > 0, f\ (x) < g(x)$, since $\dfrac{f(x)}{g(x)} = e^{-3x/2} < 1$; and

ii) if $x < 0, f\ (x) > g(x)$, since $\dfrac{f(x)}{g(x)} = e^{-3x/2} > 1$.

Moreover, $f(x), g(x) > 0$; we can deduce that the draw of A will be approximately like,



Then, Pareto order describes the set properties:

maximum of (A), minimum of (A) don't exist.

maximals of $(A) = \{(x, f(x)) : -1 \leq x \leq 0\} \cup \{(x, g(x)) : 0 \leq x \leq 1\}$.

minimals of $(A) = \{(x, g(x)) : -1 \leq x \leq 0\} \cup \{(x, f(x)) : 0 \leq x \leq 1\}$.

(b) First of all, looking at the position of the graphs we know that:

area(A)$= \int_{-1}^{0} (e^{-2x} - e^{-x/2})dx + \int_{0}^{1} (e^{-x/2} - e^{-2x})dx =$

$= [-e^{-2x}/2 + 2e^{-x/2}]_{-1}^{0} + [-2e^{-x/2} + e^{-2x}/2]_0^1 =$

$= [-1/2 + 2 + e^2/2 - 2e^{1/2}] + [-2e^{-1/2} + e^{-2}/2 + 2 - 1/2] =$

$= 3 + e^2/2 - 2e^{1/2} - 2e^{-1/2} + e^{-2}/2$ area units.