Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	6	Total
Points							

Department of Economics. **Mathematics I. Final Exam** **June 25th 2021**

Exam time: 2 hours.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function** $f(x) = \ln(ex - x^2)$**. Then:**

(a) find the asymptotes of the function and the intervals where $f(x)$ increases and decreases.

(b) find the local and/or global maximum and minimum, and range (or image) of $f(x)$. Draw the graph of the function.

(c) Consider $f_1(x)$ to be the function $f(x)$ defined on the interval where it is increasing, sketch the graph of the inverse function of $f_1(x)$.
(*Hint for the graphs:* use that $1 < \ln 4 < 2$).

0.4 points part a); 0.4 points part b); 0.2 points part c)

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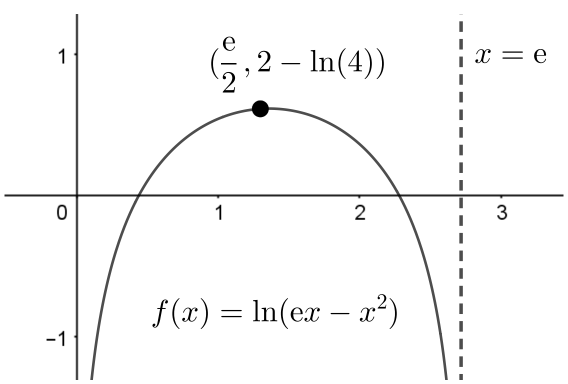
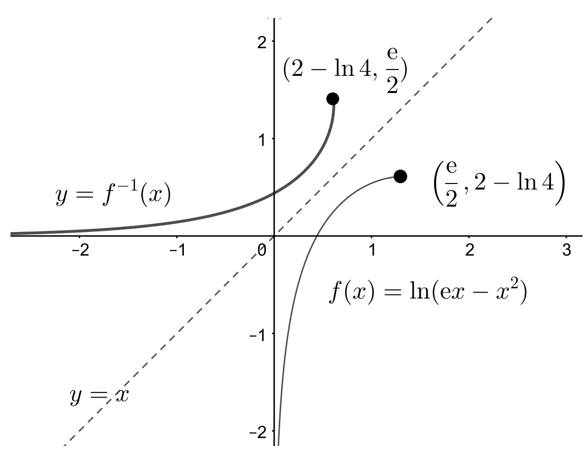
a) The domain of the funcion is $(0, e)$.
Since f is continuous on its domain, a bounded set, it is only needed to study the asymptotes at 0 and e:
i) $\lim\limits_{x\longrightarrow 0^+} f(x) = \ln(0^+) = -\infty$, $\lim\limits_{x\longrightarrow e^-} f(x) = \ln(0^+) = -\infty$.
Therefore $f(x)$ has a vertical asymptote at $x = 0$ and $x = e$.
Moreover, as $f\,'(x) = \dfrac{e-2x}{ex-x^2}$, we can deduce: f is increasing $\Longleftrightarrow$
$\Longleftrightarrow f\,'(x) > 0 \Longleftrightarrow e - 2x > 0$;
then f is increasing on $(0, \frac{e}{2}]$. Analogously, f is decreasing on $[\frac{e}{2}, e)$.

b) Interpreting the monotonicity of f, it is deduced that there are not local or global minimizers and $x = \frac{e}{2}$ is the only local and global maximizer.
Finally, as $f(\frac{e}{2}) = \ln(e^2/4) = 2 - \ln 4$ and $\lim\limits_{x\longrightarrow 0^+} f(x) = -\infty$ due to the Intermediate Value Theorem we can deduce that the range of the function will be $(-\infty, 2 - \ln 4]$.
The graph of f will have an appearance approximately, similar to the one in figure A.

c) We know that, f_1 is increasing on $(0, \frac{e}{2}]$, $f_1(0^+) = -\infty$, $f_1(\frac{e}{2}) = 2 - \ln 4$. Therefore, the graph of its inverse will be increasing as well, $\lim\limits_{x\longrightarrow -\infty} f_1^{-1}(x) = 0^+$, $f_1^{-1}(2 - \ln 4) = \frac{e}{2}$.
Therefore, the graph of its inverse will have an appearance approximately, similar to the one in figure B:



(A)



(B)

(2) **Given the implicit function $y = f(x)$, defined by the equation $e^{xy} + x^2 + y^2 = 5$ in a neighbourhood of the point $x = 2, y = 0$, it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function at $a = 2$.

(b) sketch the graph of the function f near the point $x = 2,\ y = 0$.
Use the tangent line to the graph of $f(x)$ to obtain the approximate values of $f(1.9)$ and $f(2.1)$.

(c) Will $f(2)$ be greater, less or equal than the exact value of $\frac{1}{2}(f(1.9) + f(2.1))$?
(*Hint for part (c):* use that $f''(2) < 0$).
0.4 points part a); 0.3 points part b) 0.3 points part c)

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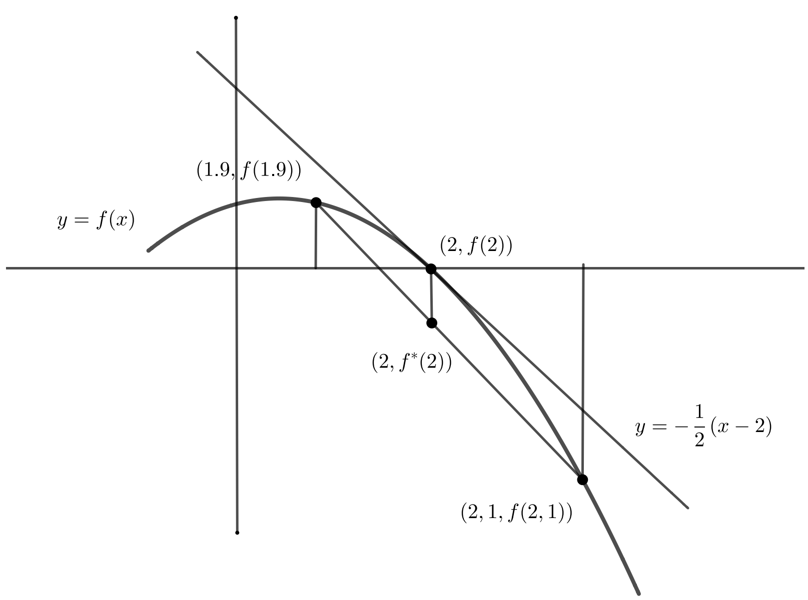
a) First of all, we calculate the first-order derivative of the equation:
$e^{xy}(y + xy') + 2x + 2yy' = 0$
evaluating at $x = 2,\ y = 0$ we obtain: $2y' + 4 = 0 \Longrightarrow y'(2) = f\ '(2) = -2$.
Then the equation of the tangent line is: $y = P_1(x) = -2(x - 2)$.
Secondly, we calculate the second-order derivative of the equation:
$e^{xy}(y + xy')^2 + e^{xy}(2y' + xy'') + 2 + 2(y')^2 + 2yy'' = 0$
evaluating at $x = 2, y = 0, y' = -2$ we obtain $y''(2) = f''(2) = -11$.
Therefore, the second-order Taylor Polynomial is: $y = P_2(x) = -2(x - 2) - \frac{11}{2}(x - 2)^2$

b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 2$ will be as you can see in the figure underneath. On the other hand, using the tangent line, the first order approximation will be:
$f(1.9) \approx -2(-0.1) = 0.2; f(2.1) \approx -2(0.1) = -0.2$.

c) Finally, since $f(x)$ is concave near $x = 2$, $\frac{1}{2}(f(1.9) + f(2.1))$ will be less than $f(2) = 0$, as you can notice looking at the graph below or if you prefer we can calculate its approximate value using the second-order Taylor Polynomial of $f(1.9)$ y $f(2.1)$); $\frac{1}{2}(f(1.9) + f(2.1)) \approx -\frac{11}{2} \cdot 0.1^2$.
Naming $f^*(2) = \frac{1}{2}(f(1.9) + f(2.1))$, the graph will be:



(3) **Let** $C(x) = \sqrt{5x^2 - 6x + 9}$ **be the cost function of a monopolistic firm, where** $x \geq 1$ **represents the quantity in kilograms of the goods. Then:**

(a) find the tangent line of $C(x)$ at $x = 3$, and obtain the approximate value of $C(3.1)$.

(b) **suppose now that** $p(x) = 29 - bx^2$ **is the inverse demand function** with $b \neq 1$, and b very close to 1.
If we know that in the previous period the firm produced 3 units, will the firm increase or decrease its production in this period?
0.5 points part a); 0.5 points part b)

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a) First of all, we calculate the first-order derivative of the equation: $C'(x) = \dfrac{10x - 6}{2\sqrt{5x^2 - 6x + 9}}$, so
$C'(3) = \dfrac{24}{2\sqrt{36}} = 2.$
Secondly, since $C(3) = 6$, the equation of the tangent line will be:
$y = 6 + 2(x - 3)$
If we approximate the value of $C(3.1)$ using the tangent line, we obtain:
$C(3.1) \approx 6 + 2(3.1 - 3) = 6.2$ monetary units.

b) The profits of the monopolistic firm are:
$B(x) = (29 - bx^2)x - C(x)$ then,
$B'(x) = 29 - 3bx^2 - C'(x) \Longrightarrow B'(3) = 29 - 27b - C'(3) = 27(1 - b).$
Thus, we have two different cases:
i) if $b < 1$, then the firm would like to increase its production.
ii)but if, $b > 1$ then the firm would reduce its production.

SOLUTIONS ANNEX FOR PROBLEMS 1, 2 AND 3.

(4) **Let $f(x) = x^4 - 2x^2$, you are asked:**

(a) state Bolzano's Theorem (Intermediate Zero Theorem) for the function g defined in the interval $[a, b]$.

(b) suppose that $a = -2$. Find b such that $f(x)$ satisfies the hypothesis of this theorem.

(c) suppose that $a = -2$. Find b such that $f(x)$ satisfies the thesis of this theorem.

(*Hint for part b) and c):* sketch the graph of the function)

0.2 points part a); 0.4 points part b); 0.4 points part c)

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a) Bolzanos' Theorem states that if g is continuous on the interval $[a, b]$, and the function satisfies that

i) $g(a) < 0 < g(b)$; or

ii) $g(b) < 0 < g(a)$

then there exists a point c in (a, b) such that $g(c) = 0$.

b) since $f(-2) > 0$, then b must satisfy $f(b) < 0$; this is, $b \in (-\sqrt{2}, 0) \cup (0, \sqrt{2})$.

c) It is needed a zero of the given polynomial in the interval (a, b).

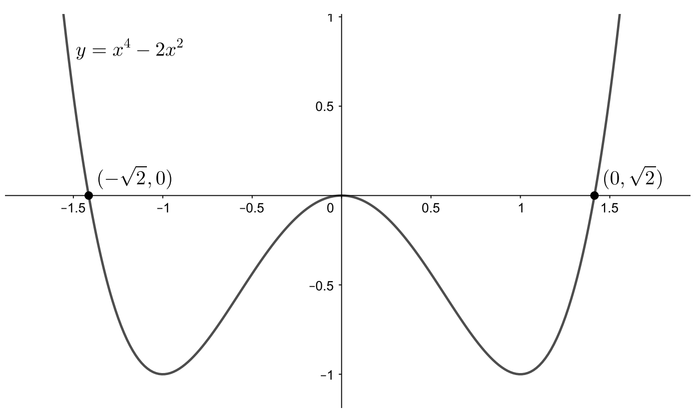
Thus, $b \in (-\sqrt{2}, \infty)$.

Notice: since $f(x) = x^2(x - \sqrt{2})(x + \sqrt{2})$, we can deduce:

i) $f(x) < 0$ if $x \in (-\sqrt{2}, 0) \cup (0, \sqrt{2})$;

ii) $f(x) > 0$ if $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

Therefore, the graph of f will be approximately:



(5) **Given the functions** $f, g : [-1, 1] \longrightarrow \mathbb{R}$, **defined by:** $f(x) = -e^{2x}, g(x) = \dfrac{3}{2+x}$, **then:**

(a) draw approximately the set A, bounded by the graph of these functions and the straight lines $x = -1$, $x = 1$. Find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) calculate the area of the given set.

Hint for part (a): Pareto order is defined as: $(x_0, y_0) \leq_P (x_1, y_1) \Longleftrightarrow x_0 \leq x_1, y_0 \leq y_1$.

0.6 points part a); 0.4 points part b).

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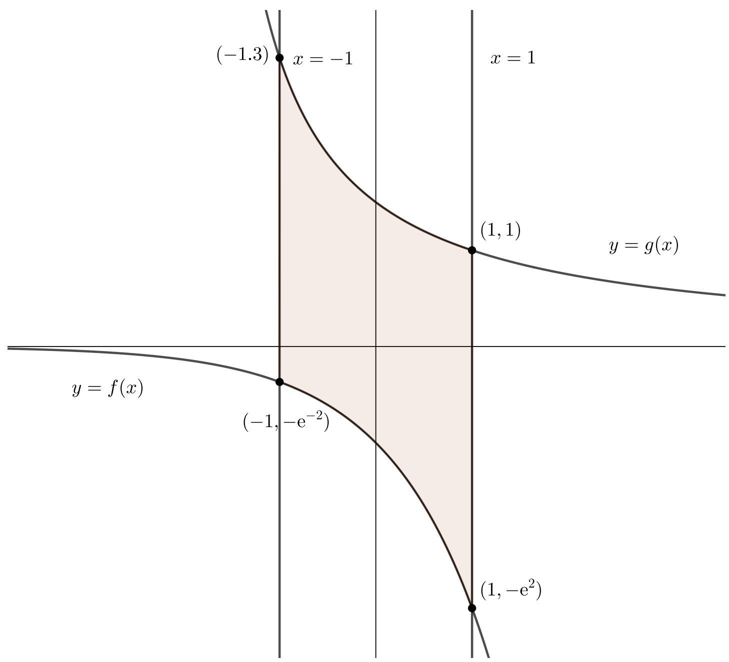
a) $f(x)$ and $g(x)$ are deceasing. Furthermore, since $f(x) < 0 < g(x)$, the draw of A will be approximately like,



Then, Pareto order describes the set properties:
maximum of (A), minimum of (A) don't exist.
minimals of $(A) = \{(x, f(x)) : x \in [-1, 1]\}$
maximals of $(A) = \{(x, g(x)) : x \in [-1, 1]\}$

b) First of all, looking at the position of the graphs we know that:

area(A)$= \int_{-1}^{1} (g(x) - f(x))dx = \int_{-1}^{1} (\dfrac{3}{2+x} + e^{-2x})dx =$

(applying Barrow's Rule we obtain:)

$= [3\ln(2+x) + \frac{1}{2}e^{2x}]_{-1}^{1} = (3\ln 3 + \frac{1}{2}e^2) - (0 + \frac{1}{2}e^{-2}) =$

$= 3\ln 3 + \frac{1}{2}e^2 - \frac{1}{2}e^{-2}$ area units.

(6) **Given the function** $f(x) = \dfrac{x}{\sqrt{x+1}}$, **defined on** $(-1, \infty)$. **Then:**

(a) Find the equation of the primitive function $F(x)$ of $f(x)$ such that $F(0) = \frac{2}{3}$.

(b) Study the concavity, convexity and the inflexion point of the previous primitive function $F(x)$.

Hint for part b): for this part you don't need to solve part a) above.

(c) What can you say about $F(x)$ at the point $x = 0$?

0.4 points part a); 0.4 points part b); 0.2 points part c)

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a) Using the change of variable $x + 1 = t^2, dx = 2tdt$, we can obtain the primitive function of f, it will be:

$F(x) = \displaystyle\int \frac{x}{\sqrt{x+1}} dx = \int \frac{t^2 - 1}{t} 2tdt = 2 \int (t^2 - 1)dt = 2(t^3/3 - t) + C =$

$= \frac{2}{3}\sqrt{(x+1)^3} - 2\sqrt{x+1} + C$

Since $\frac{2}{3} = F(0) = \frac{2}{3} - 2 + C$, we can deduce that $C = 2$.

The primitve can be obtained as well:

- integrating by parts considering $f = x$, $g' = \frac{1}{\sqrt{x+1}}$, and therefore $g = 2\sqrt{x+1}$
- with the sustitution $x + 1 = t$
- by adding $+1 - 1$ in the numerator

b) i) First of all, $F'(x) = f(x)$, using The Fundamental Theorem of Calculus.

ii) secondly, $F''(x) = f'(x) = \dfrac{\sqrt{x+1} - x\frac{1}{2\sqrt{x+1}}}{(\sqrt{x+1})^2} = \dfrac{x+2}{2\sqrt{x+1}(x+1)}$

iii) then, $F''(x) > 0 \iff x + 2 > 0 \iff x > -2$ and therefore, $F(x)$ is a convex function in its domain $(-1, \infty]$.

c) Moreover, since F is convex in its domains, it is going to attain its global minimum at its critical point $x = 0$ $(F'(0) = 0)$.

SOLUTIONS ANNEX FOR PROBLEMS 4, 5 AND 6.